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The propagation of a crack surrounded by damage is simplistically replaced by that of a crack lodged in a vanishingly thin or small elastic inhomogeneity. This latter problem is asymptotically analyzed and numerically solved via the use of Fast Fourier Transform Algorithm.

A re-examination of Mindlin's grade-3 elasticity is thoroughly carried out to reveal the interaction between mechanical loading and surface tension.

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FINAL REPORT

Eshelby Forces Associated with An Advancing Crack Surrounded by Vanishingly Small Inhomogeneity

by

Chien H. Wu
University of Illinois at Chicago
May, 1991

A-1

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1. Introduction

When a specimen with an edge crack is subjected to fatigue loading, anisotropic and inhomogeneous damage will slowly develop and eventually evolve into a typical configuration in which a crack of length unity is lodged in a vanishingly small crack–tip damage. Similarly when a notched specimen is fatigue loaded, a crack completely surrounded by damage will eventually emerge. These two cases are depicted in Figs. 1 and 2 where $\epsilon<<1$ and $r_o(\theta)$ defines the shape of the damage zone.

Fig. 1

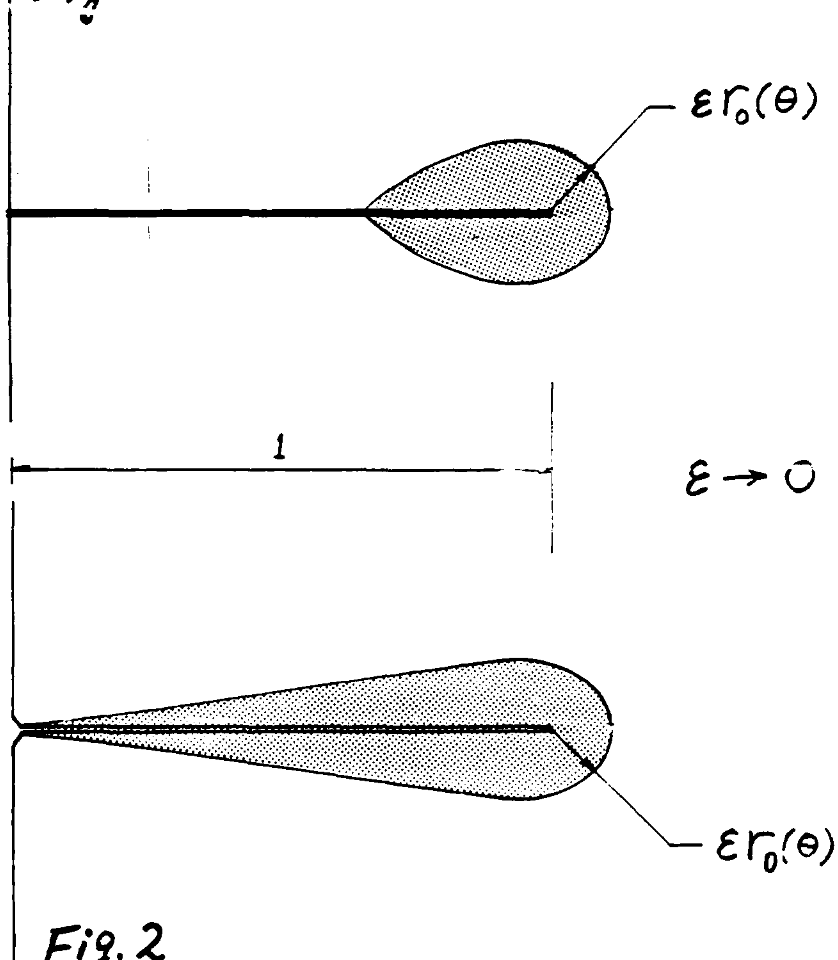


Fig. 2

In order to understand the many fundamental physical phenomena associated with the indicated crack–damage evolution, a meaningful description of the so–called damage must first be given. Then the kinetic relation associated with the damage evolution must also be fully explored and developed. Both issues have attracted the attention of many researchers for many years, but the results have been rather inconclusive. It is partly for this reason that neither issue was considered in a direct way in this project. Instead we took an extremely simplistic approach by assuming that the main characteristics of a damaged material is that it is softer than the original material. This assumption reduces the unknown damage to an inhomogeneity that may be characterized by two elastic constants which may in turn be interpreted as damage parameters.

When a crack is interacting with an inhomogeneity, such as the cases depicted in Figs. 1 and 2, the associated elasticity problem is well defined and hence a full stress analysis is possible. The availability of this solution makes it feasible to link the Eshelby (configuration) forces to the inhomogeneity geometry and moduli, which are used as a gross representation for the damage. Thus, devising an efficient numerical procedure for obtaining the desired elasticity solution became a main objective of this one–year project. This objective was completed and the relevant results may be found in Chao–Hsun Chen's thesis which was completed in 1990 under the supervision of the PI (Appendix I).

It is perhaps clear from the general nature of the problem that the desired solution can only be obtained by numerical means. However, the actual computation must be preceeded by a skillful asymptotic analysis so that the $\epsilon \to 0$ limit is analytically factored out of the ensuing calculation. This is necessary because no numerical scheme can possibly handle the conflicting limits required by $\epsilon \to 0$ (almost no inhomogeneity) and $r^{-1/2} \to \infty$ (crack–tip inside the inhomogeneity). Such an asymptotic analysis, together with the accompanying computational scheme, was successfully accomplished.

A more detailed description of this accomplishment is summarized in Section 2.

The above objective bypasses the need and difficulty of addressing the many physical phenomena associated with a typical state of damage, and yet the defining of a state of damage, together with the associated kinetics, remains to be the most crucial issue. It is with this consideration in mind that we chose to re–examine the implications of higher order theories and found that the well–established grade–3 theory is the most outstanding candidate. This accomplishment is summarized in Section 3.

2. Numerical Solution of a Crack Interacting with an Inhomogeneity

The elasticity problems considered are those depicted in Figs. 1 and 2 where the damaged material is simplistically replaced by a different elastic material, and is henceforth termed an inhomogeneity. The geometry of a typical problem is therefore defined by a dimensionless half–crack length 1, an inhomogeneity denoted by D_1 which in term is embedded in a medium denoted by D_2. The associated elasticity problems are formulated in terms of complex functions.

Let (z_1,z_2) be rectangular Cartesian coordinates and $z=z_1 + iz_2$ the associated complex variable in the z–plane. For plane problems, the displacements $u_\alpha(z_1,z_2)$, stresses $\tau_{\alpha\beta}(z_1,z_2)$ and resultant force over an arc $R = R_1 + iR_2$ may be expressed in terms of two complex functions $W(z)$ and $w(z)$ viz.

$$2\mu(u_1+u_2) = \kappa W(z) - z\overline{W'(z)} - \overline{w(z)}$$

$$iR = W(z) + z\,\overline{W'(z)} + \overline{w(z)}$$

where

$$\kappa = \begin{cases} 3-4\nu & \text{plane stress} \\ (3-\nu)/(1+\nu) & \text{plane strain} \end{cases}$$

and μ and ν are, respectively, shear modulus and Poisson's ratio. For the class of problems under consideration, the solution is more conveniently expressed in terms of W and another complex function f defined by

$$f(z) = W(z) - z\,\overline{W'(\bar{z})} - \overline{w(\bar{z})}$$

As a convention, an additional subscript α is placed on a parameter or variable to indicate its region of definition D_α. Thus, W_α and f_α are defined on D_α for which the elastic constants are μ_α, ν_α and κ_α. Finally, for a two-component composite the following composite parameters are important constants

$$\gamma = \frac{(1+\kappa_2)\mu_1}{(1+\kappa_1)\mu_2}\ ,\ \ \gamma^* = \frac{1}{1+\kappa_1}\left[1-\frac{\mu_1}{\mu_2}\right]$$

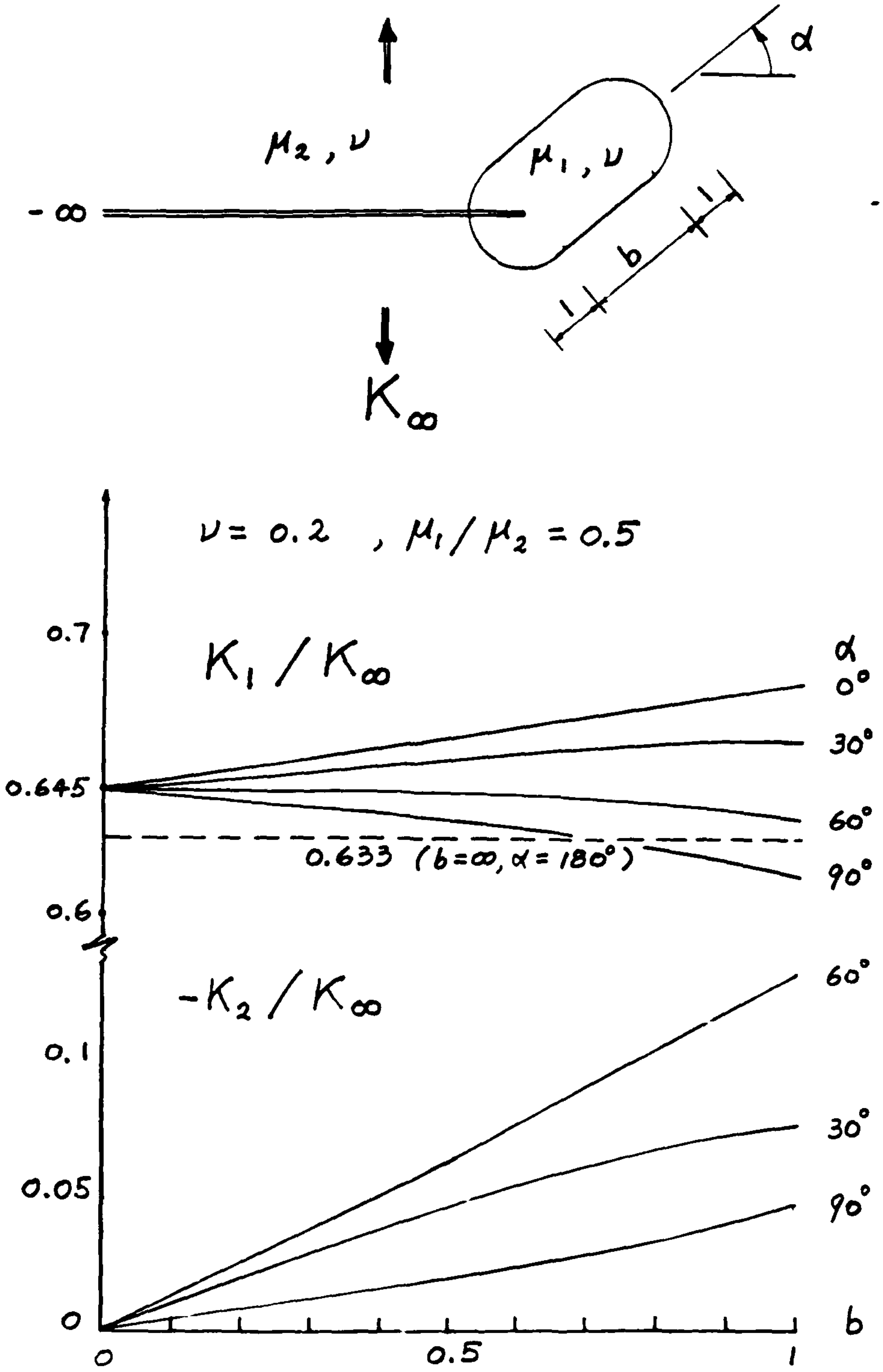
where μ, κ_1 are associated with the inhomogeneity.

For a given problem the four complex functions W_1, f_1, W_2 and f_2 are obtained as infinite series. The constant coefficients are then determined by employing the readily available Fast Fourier Transform Algorithm.

A) A Crack Lodged in a Tip Inhomogeneity.

The relevant analysis and results may be found in Chapter III of Appendix I. The asymptotic analysis is thoroughly carried out to the order of ϵ, and the final set of equations to be solved numerically are given by equations (3.21) and (3.22) of Appendix I. The implementation of Fast Fourier Transform subroutines is illustrated by equation (3.28).

Our scheme appears to be more powerful than most of the known techniques in that solutions accurate to the order of ϵ can actually be computed (c.f. (3.29)). Moreover, nonsymmetric problems can be handled just as easily as symmetric ones. The set of results associated with Figs. 3.1 and 3.2 of Appendix I shows that the scheme is not affected by the slenderness of the inhomogeneity (a circula imhomogeneity is the least troublesome in every respect). To illustrate the efficiency of our scheme for nonsymmetric configurations, a slender inhomogeneity inclined at an angle α from the main crack is used as an example, and no convergence difficulties were encountered in the calculation (Fig. 3).



μ_2, ν
μ_1, ν
$-\infty$
α
b
1
1
K_∞
$\nu = 0.2$, $\mu_1/\mu_2 = 0.5$
K_1/K_∞
α
0°
30°
60°
90°
0.7
0.645
0.6
0.633 ($b=\infty, \alpha=180°$)
$-K_2/K_\infty$
60°
30°
90°
0.1
0.05
0
b
0
0.5
1

Fig. 3

(B) A Crack Surround by a Thin Inhomogeneity.

The relevant analysis and results may be found in section 4.4 of Appendix I. One of the presumed properties of the thin inhomogeneity is that it shrinks onto the crack as $\epsilon \to 0$.

The most important and new analytic conclusion of our investigation is the property of the solution for D_2 in the immediate neighborhood of a crack tip. It is shown that the solution must involve powers of $r^{-1/2}$ where $r = 0$ is the crack tip which is in D_1. This is in contrast to the crack–tip inhomogeneity solution where the series is of the form of $r^{-(n + 1/2)}$.

This derived property establishes the form of the desired series solution upon which the Fast Fourier Transform Algorithm is again applied. The accuracy and efficiency (in terms of the number of terms needed) of the scheme are checked by the special case of a confocal ellipse for which a numerically exact solution is available (Figs. 4.2.3, 4.2.4 and 4.3.2 of Appendix I).

(C) Eshelby Tensor and Eshelby (Configuration) Forces.

The relevant analysis are carried out in detail in Chapter V of Appendix I. Let $f(\epsilon_{\alpha\beta})$ be the strain energy density function, so that

$$\tau_{\alpha\beta} = \partial f / \partial \epsilon_{\alpha\beta} .$$

The Eshelby tensor $p_{\beta\alpha}$ is defined by

$$p_{\beta\alpha} = f\delta_{\beta\alpha} - \tau_{\beta\gamma} u_{\gamma,\alpha} .$$

It is known that the various configuration forces, which are the generalized forces associated with damage evolution, are but integrated forms of $p_{\beta\alpha}$. The calculation of the many configuration forces would be greatly facilitated by a suitable representation of $p_{\beta\alpha}$.

This objective is realized in Chapter V of Appendix I. It is shown that

$$p_{12} - p_{21} = 2(W^{*\prime} + \overline{W^{*\prime}})$$

$$(p_{12} + p_{21}) + i2p_{22} = -2(z\overline{W^{*\prime}} + \overline{w^{*\prime}})$$

where w^* and W^* are two new complex functions defined by

$$w^{*\prime}(z) = -\frac{\kappa+1}{4\mu}\, i(W'(z))^2$$

$$w^{*\prime}(z) = -\frac{\kappa+1}{4\mu}\, i2W'(z)\, w'(z) \ .$$

Since both W and w are already in series form, the desired convenient representation is complete.

3. Grade–3 Elasticity and Surface Phenomena

The issue of damage evolution was deliberately bypassed in the previous section for the simple reason that there has not been to date a complete and definitive continuum theory that may be used to study fatigue damage propagation. At the same time it is clear that damage leads to failure and the creation of new surfaces. It is therefore desirable to look for a theory that actually includes surface tension as one of its variables. The grade–3 elasticity, which was fully developed by Mindlin in 1965, has just such a variable. A detailed re–examination of the theory, with special emphasis on surface phenomena, is submitted as the second accomplishment of this one–year project. The main results are delineated in a manuscript which is attached as Appendix II.

In view of the complexity of the theory, none of the mathematical deductions and formulae are reproduced in this section. Highlights of our accomplishment include the specific determination of the energy associated with a surface, with or without the presence of body forces and/or other surface tractions. These results are presented in detail in a self contained manner in Sections 6 and 7 of Appendix II.

The deduction given in Section 4 of Appendix II has effectively reduced the complexity of the theory to an extent that only a series of much simpler problems of familiar properties needs to be solved. In view of the exploratory nature of this inestigation, this accomplishment is most significant in that it can be readily applied to yield solutions to important benchmark problems upon which the physical significance of

the theory could be meaningfully evaluated. Point load, cracks and notches are among the first ones to be continually examined by us.

4. Written Publications, Degrees Awarded and Presentations.

A. Written Publications

(a) Chen, Chao–Hsun, "Cracks in Vanishingly Thin Inhomogeneities and the Associated Energy Release Rates", Ph.D. Thesis, University of Illinois at Chicago, 1990.

(b) Wu, Chien H., "Cohesive Elasticity and Surface Phenomena", Quarterly of Applied Mathematics, to appear.

B. Degrees Awarded

Chen, Chao–Hsun, Ph.D. University of Illinois at Chicago, 1990.

Dr. Chen is now an assistant professor of mechanics at the Institute of Applied Mechanics, National Taiwan University.

C. Conference Presentation

Wu, Chien H. "Adhesion, Interfacial Energy and Failure of a Grade–2 Elastic String, presented at the International Symposium" on Micromechanics: Homogenization, Heterogenization and Strength, March, 1991.

Appendix I

Cracks in Vanishingly Thin Inhomogeneities and the Associated Energy Release Rates

by

Chao–Hsun Chen

Ph.D. Thesis
University of Illinois at Chicago
1990

Advisor: Chien H. Wu

TABLES OF CONTENTS

CHAPTER			PAGE
0		ABSTRACT	4
I		INTRODUCTION	5
II		COMPLEX VARIABLE FORMULATION	10
III		A CRACK IN THIN DISJOINTED INHOMOGENEITIES	14
	3.1	Introduction	14
	3.2	Formulation	15
	3.3	Outer Expansion	16
	3.4	Inner Expansion	17
	3.5	Fast Fourier Transform Algorithm and Numerical Procedure	19
	3.6	Results and Discussions	22
IV		A CRACK IN A THIN INHOMOGENEITY	26
	4.1	Introduction	26
	4.2	Series Solution	28
	4.2.1	Introduction	28
	4.2.2	Formulation	29
	4.2.3	Anti-Plane Shear	31
	4.2.4	Plane Problems	33
		i. Exact Solution for Equal Shear Modulus	35
		ii. Series Solution	37
		iii. Very Large Inhomogeneity	39
	4.3	Asymptotic Solution	46
	4.3.1	Formulation	46
	4.3.2	Outer Expansion	47
	4.3.3	Inner Expansion	52
	4.3.4	An Approximate Analytic Solution	58
	4.3.5	Results and Discussion	62
	4.4	A Crack in a Thin Inhomogeneity of Arbitrary Shape	68
	4.4.1	Introduction	68
	4.4.2	Formulation	69
	4.4.3	Outer Expansion	69
	4.4.4	Inner Expansion	71
	4.4.5	Results and Discussion	74
V		ESHELBY TENSOR AND THE ASSOCIATED ENERGY RELEASE RATES	75
	5.1	Plane Elasticity – The τ – Problem	75
	5.2	The p – Problem and τ^* – Problem	78
	5.3	Eshelby Tensor Associated with A τ – Problem	80

VI SUMMARY 85

REFERENCES 87

LIST OF FIGURES

FIGURE		PAGE
1.1	A crack with tips lodged in Vanishingly small inhomogeneities	9
1.2	A crack in a thin inhomogeneity	9
3.1	Configuration of a crack with a tip inhomogeneity of thickness 2 and length (2+a+b)	24
3.2	K_1 as a function of b with a as a parameter	25
4.2.1	A crack in a confocall elliptic inhomogeneity a) Configuration in the physical z–plane b) The image of the crack–inhomogeneity configuration in the ζ–plane	42
4.2.2	K_3 as a function of μ_1/μ_2	43
4.2.3	K_1 as a function of μ_1/μ_2 with ρ_o as a parameter	44
4.2.4	K_1 as a function of ρ_o^{-2} with μ_1/μ_2 as a parameter	45
4.3.1	A crack in a vanishingly thin elliptic inhomogeneity a) Configuration in the physical z–plane $b/a \to 0$ b) Parabolic nose in the boundary layer ζ–plane	64
4.3.2	K_1 as a function of μ_1/μ_2 for $b/a \to 0$ and $\nu_1 = \nu_2 = 0.2$ (plane strain)	65
4.3.3	Admissible region in terms of composite elastic constants (α,β) and (γ,γ^*)	66
4.3.4	K_1 and K_2 for all admissible combinations of γ and γ^*	67
6.1	K_1 as a function of μ_1/μ_2 for $\nu_1 = \nu_2 = 0.2$. The five inhomogeneity configurations are : 1) Very large confocal ellipse 2) The shape of Fig. 3.1 3) Crack–tip circle 4) Vanishingly thin confocal ellipse 5) Semi–Infinite crack perpendicular to a two phase boundary	86

ABSTRACT

Newly engineered high-performance composite materials are very often reinforced by particles, continuous or short fibers and thin layers. Cracks encountered in such materials are more often than not affected by its tip being located in one particular small particle or thin layer of a composite. The physical effect of such apparently small geometric alterations on the toughness of the material is finite and must be carefully examined.

Fatigue crack propagation usually leads to the formation of a thin layer of damaged material surrounding the propagating crack. The thin layer of damage, however, is known to have finite effects on the various generalized Eshelby forces that drive the damage.

The interaction of a crack and a small inhomogeneity in an otherwise homogeneous medium is studied in this thesis. Asymptotically deduced computer codes are developed for the purpose of computing any and all physical quantities relevant to the aforementioned problems.

CHAPTER I

INTRODUCTION

With the advent of engineered multiphase materials in recent years, most notably composites reinforced by particles, continuous or short fibers and thin layers, there has come an interest in crack problems involving bodies of spatially varying material properties. One particular aspect of the problem is the alteration in the stress intensity factor (SIF) from its apparent value due to either the crack tip being lodged in a region with elastic moduli differing from the bulk or the complete crack being located in one particular phase of a composite. A crack partially penetrating a thin fiber and a crack situated inside of a thin layer are but two examples. It is noted that the word thin is appended to stress the very often encounted physical situation. The results presented in this thesis have direct bearing on the understanding of the class of problems.

Fracture toughness enhancement has been observed in a number of ceramic systems containing particles which undergo a transformation of the martensite type (T.K. Gupta etal 1978). The high stresses in the vicinity of a macroscopic crack induce a transformation of the particles and thereby alter the crack tip stress field. The associated SIF could be approximated by that for a crack tip lodged in a thin inhomogeneity with moduli softer than the bulk.

It is now a common knowledge that there exists a fracture process region or damage zone near a crack tip where fracture process such as nucleation of voids or microcracks and their coalescence take place and the usual continuum theory does not apply. The damage zone is usually small compared with the length of the crack and may be approximated by a thin inhomogeneity with softer moduli.

The size and shape of a process region change as it moves along with the crack tip

under fatigue loading conditions. The net result is that of a crack surrounded by an active crack–tip process region together with an inactive wake, the damage region created and left behind by the traversing active region. This is just the crack–layer configuration investigated by Chuduovsky, Moet and Botsis (1987). If we approximate the damaged region by an elastic material with softer moduli, the crack–layer configuration is just that of a crack lying inside of a thin inhomogeneity embedded in an otherwise homogeneous medium. The change of the crack–layer configuration leads to a number of identifiable energe–release rates which cannot be determined withour a careful stress analysis. The results presented in this thesis provide effective means to perform the needed calculation.

With the above problems and the attending importance and applications in mind, we direct our attention in this thesis to the following two specific classes of problems:

Problem I . Disjointed Inhomogeneities

A crack of length 2 in an infinite plane is assumed to have its tips separately lodged in vanishingly small inhomogeneities of size ϵ ($\epsilon << 1$), Fig. 1.1.

Problem II . Single Inhomogeneity

A crack of length 2 in an infinite plane is wholly embeded inside of an inhomogeneity of vanishingly small thickness ϵ ($\epsilon << 1$), Fig. 1.2

The choice of the basic configuration, a straight crack in an infinite plane, enables us to remove the geometric and loading complications, which may be handled by conventional means, from the new and essential asymptotic deduction as well as the appending numerical scheme. In fact, the ultimate objective is to incorporate the asymptotic/numerical result of the thesis into a general situation for practical applications.

Both the inhomogeneity and the infinite medium are assumed to be homogeneous and isotropic in this thesis. The analysis may be straightforwardly extended to cases where the thin inhomogeneity is anisotropic, but detailed calculations are not included in this thesis. Similar approach could be applied to the situation where the inhomogeneity is actually inhomogeneous. This latter consideration requires extensive analysis and is not considered.

The solutions to the two classes of problems depend, among other parameters, on the size of the inhomogeneity, i.e., the small parameter ϵ. It is clear from the general nature of the problems that no analytic and explicit solutions can be expected. The implementation of a computational scheme, however, must be preceded by an asymptotic analysis to get rid of the ϵ, as no numerical scheme can possibly handle the conflicting limits required by $\epsilon \to 0$ (almost no inhomogeneity) and $r^{-1/2} \to \infty$ (crack tip inside of the vanishingly small/thin inhomogeneity). Such an asymptotic analysis, together with the attending numerical scheme, represents the main original contribution of this thesis. The final product are two computer codes which, together with other codes that may be developed to accommodate geometric and loading conditions, may be immediately adapted for application.

To ensure the correctness and accuracy of the computer codes, a number of benchmark problems are also considered. Some of them are also new and original but the main motivation was for the purpose of validating the anticipated computer codes.

For problem I, both the inhomogeneity and the medium contain a traction–free boundary and the analytical structure of the solution follow directly from that of the well–known crack–tip field representation. The standard Fast Fourier Transform routines are used to determine the unknown coefficients in the series representation, and the capability of generating the solution to the order of ϵ is established. The benchmark problem is that of a semi–infinite crack penetrating a circular inhomogeneity, a numerically exact solution obtained by Steif(1987).

For problem II, the medium contains no portion of the traction–free boundary and, as a result, the analytical structure of the solution for the medium in the vicinity of a tip is not clear. For this reason, the case of a crack embeded inside of a confocal elliptic inhomogeneity is introduced as a benchmark problem and studied in detail. The confocal gemetry accommodates a Fourier series representation but the convergence of the series becomes extremely slow as the inhomogeneity becomes vanishingly thin. Nevertheless the

$\epsilon \to 0$ limit is established by extrapolation and a numerically exact benchmark is established. The aforementioned analytic structure, however, is still unknown. This important information is revealed by a detailed asymptotic analysis.

The required asymptotic analysis parallels to that used in thin airfoil analysis but is much more involved, as both the "airfoil" and the outside medium are each governed by two complex functions. Moreover, there is a square-root singularity inside the "airfoil". The analysis is carried out systematically and the needed analytical structure is extracted from the inner expansion of the outer expansion. This property is used to construct the needed series solution whose coefficients are again determined by employing the readily available Fast Fourier Transform routines. As an unexpected byproduct, an approximate but explicit solution is also obtained for the confocal crack problem. This result, together with the numerically exact Fourier series solution, is used to validate the final computer code.

Numerical results obtained from the computer codes developed for the two problems are further validated by the following intuitive considerations. For problem II, the center portion of the inhomogeneity is expected to have small effect on the solution as $\epsilon \to 0$. If so the solution for Problem II should tend to that for Problem I in terms of a geometrically obvious parameter. Such a tendency does appear to exist. Extending the length of the inhomogeneity of problem II would result an increase in SIF and its vallue would always be bounded from above by that for a crack embeded in an infinite layer of vanishing thickness. Our numerical results also conform with this intuitive observation.

Chapter 2 summarizes the formulation in terms of a complex variable. The rest of the exposition follows approximately the order of outlining delinedted in this chapter.

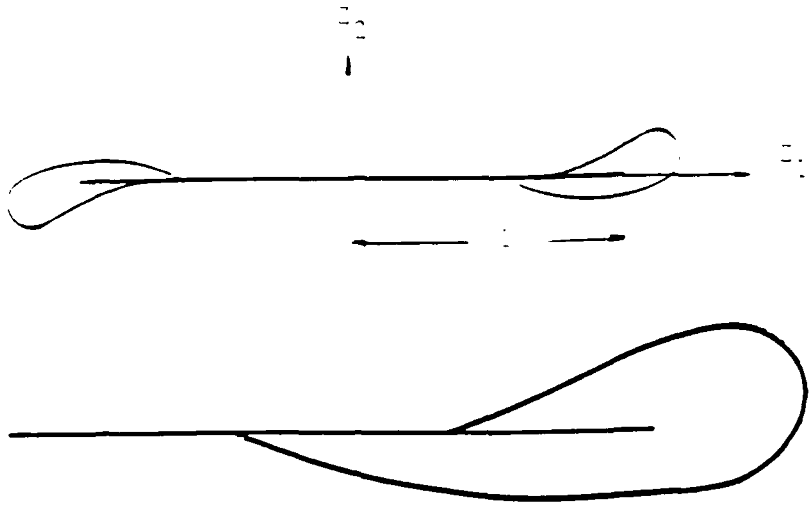


Fig. 1.1 A crack with tips lodged in Vanishingly small inhomogeneities

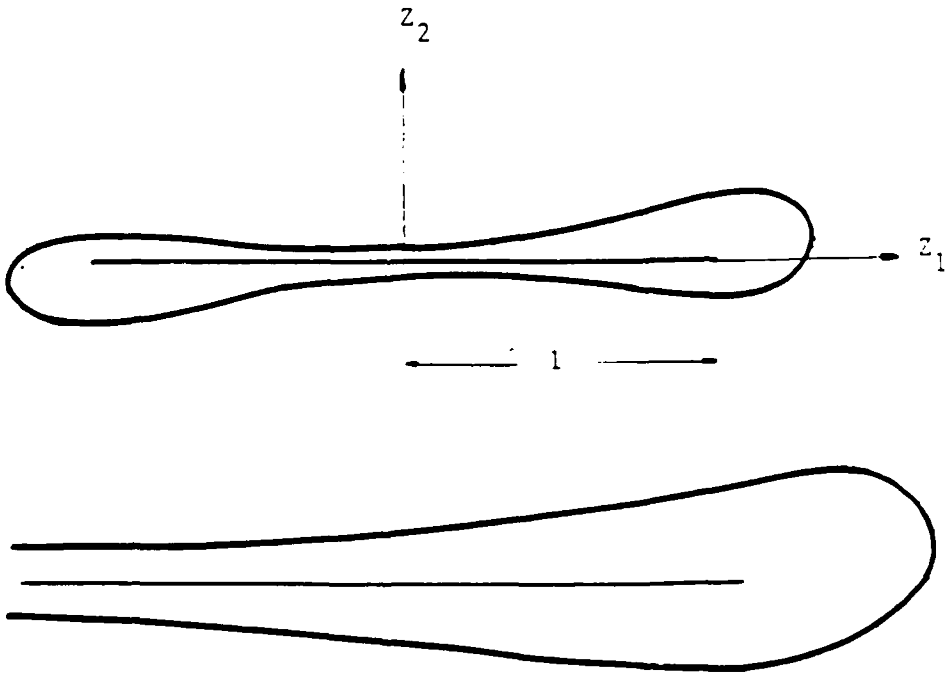


Fig. 1.2 A crack in a thin inhomogeneity

CHAPTER II

COMPLEX VARIABLE FORMULATION

Let (z_1,z_2) be rectangular Cartesian coordinates and $z = z_1 + iz_2$ the associated complex variable in the z–plane. For plane problems, the displacements $u_\alpha(z_1,z_2)$, stresses $\tau_{\alpha\beta}(z_1,z_2)$ and resultant force over an arc $R = R_1 + iR_2$ may be expressed in terms of two complex functions $W(z)$ and $w(z)$, viz.,

$$2\mu(u_1 + iu_2) = \kappa W(z) - z\overline{W'(z)} - \overline{w(z)}\,, \tag{2.1}$$

$$iR = W(z) + z\overline{W'(z)} + \overline{w(z)}\quad , \tag{2.2}$$

where

$$\kappa = \begin{cases} 3 - 4\nu & \text{plane stress} \\ (3-\nu)/(1+\nu) & \text{plane strain} \end{cases} \tag{2.3}$$

and μ and ν are, respectively, shear modulus and Poisson's ratio.

For regions containing a portion of the real axis along which displacement and traction are continuous, the function $w(z)$ may be expressed in terms of $W(z)$ and a new function $f(z)$ as follows (England, 1971):

$$w(z) = \overline{W(\bar{z})} - zW'(z) - \overline{f(\bar{z})}\ . \tag{2.4}$$

Using the above, we obtain from (2.1) and (2.2)

$$iR = W(z) + W(\bar{z}) + (z-\bar{z})\overline{W'(z)} - f(\bar{z})\,, \tag{2.5}$$

$$2\mu(u_1 + iu_2) = (\kappa+1)W(z) - W(z) + W(\bar{z}) + (z-\bar{z})\overline{W'(z)} - f(\bar{z}). \tag{2.6}$$

A traction–free crack of length 2 is located on the real axis with $|z_1| < 1$. The crack is

partially or wholly embedded in an inhomogeneity, denoted by D_1, which in turn is embedded in an infinite medium denoted by D_2, Fig. 2.2a. An additional subscript a will be placed on a parameter or variable to indicate its region of definition. Thus, μ_a, ν_a, κ_a, W_a and f_a are defined for region D_a. The crack is located in D_1 and the associated traction–free condition may be integrated once to become (c.f. (2.5))

$$W_1^{\pm}(x) + W_1^{\mp}(x) - f_1^{\mp}(x) = 0 \qquad (|x| < 1) \tag{2.7}$$

where the notation $F^{\pm}(x) = F(x \pm i0)$ has been used.

The infinite medium is loaded at infinity by $\tau_{\alpha\beta} = \sigma_{\alpha\beta}$ so that

$$W_2 = Wz\,, \quad f_2 = fz \qquad \text{as } z \to \infty \tag{2.8}$$

where

$$W = \frac{1}{4}(\sigma_{11} + \sigma_{22})\,, \quad f = \frac{1}{2}(\sigma_{11} - \sigma_{22}) + i\sigma_{12}\,, \tag{2.9}$$

and

$$\sigma = 2W - f = \sigma_{22} - i\sigma_{12} \tag{2.10}$$

is another parameter to be used in the sequel.

The interface between D_1 and D_2 is denoted by C and is defined by

$$C\colon z = z_c\,. \tag{2.11}$$

It is assumed that C is perfectly bonded so that traction and displacement are continuous along C. In particular, the traction continuity condition may be integrated once to become a continuity condition in R (c.f. 2.5). The two conditions are

$$\begin{aligned} W_1(z_c) + W_1(\bar{z}_c) + (z_c-\bar{z}_c)\overline{W_1'(z_c)} - f_1(\bar{z}_c) &= W_2(z_c) + W_2(\bar{z}_c) \\ &+ (z_c-\bar{z}_c)\overline{W_2'(z_c)} - f_2(\bar{z}_c)\,, \end{aligned} \tag{2.12}$$

$$W_1(z_c) = \gamma W_2(z_2) + \gamma^* \left[W_2(z_c) + W_2(\bar{z}_c) + (z_c-\bar{z}_c)\overline{W_2'(z_c)} - f_2(\bar{z}_c)\right], \tag{2.13}$$

where the R–continuity, (2.12), has been used in simplifying the displacement continuity conditions, (2.13), and

$$\gamma = \frac{(1+\kappa_2)\mu_1}{(1+\kappa_1)\mu_2}, \quad \gamma^* = \frac{1}{1+\kappa_1}\left[1 - \frac{\mu_1}{\mu_2}\right], \tag{2.14}$$

are two composite parameters. A discussion of composite parameters may be found in (Dundurs, 1969). Equations (2.7), (2.8), (2.12) and (2.13) constitute the governing conditions for the solution of the desired problem.

We shall be dealing with vanishingly small inhomogeneities and the following two cases will be considered.

A. **Disjointed Inhomogeneities.**

The crack tips are separately embeded in a vanishingly small inhomogeneity, Fig. 1.1a. The interface around $z=1$ is defined by

$$C: [\text{Right}] : z=z_c(s,\epsilon) = 1+\epsilon\left[x(s)+iy(s)\right], \; 0<s<s_o, \tag{2.15}$$

where s is an arc parameter and $\epsilon<<1$ a small parameter. The interface around $z=-1$ is assumed to satisfy the condition C[Left]= −C[Right].

B. **Single Inhomogeneity**

The crack is wholly embedded in a vanishingly thin inhomogeneity Fig. 1.2a. The interface C is given by

$$C\colon z = z_c(x;\epsilon) = \begin{cases} x \pm i\epsilon y^{\pm}(x) & |x| < 1 \\ \epsilon[\zeta(s) \mp i\eta(s)] & |x| \approx 1 \end{cases} \tag{2.16}$$

where s is a conveniently chosen arc parameter that may be expressed in terms of x, and $\epsilon << 1$ is again a small parameter.

The solutions to the associated boundary value problems must therefore depend on the size of the inhomogeneity ϵ. We shall use the generic symbol $F(z;\epsilon)$ to indicate the dependence of $F(z)$ on ϵ. The objective of this thesis is to obtain the asymptotic expansion of $F(z;\epsilon)$ as $\epsilon \to 0$. We note that the solution characterized by $u(z) = u_1 + iu_2$ satisfies the condition

$$u(-z) = -u(z) \tag{2.17}$$

for both cases.

CHAPTER III

A CRACK WITH TIPS LODGED IN VANISHINGLY SMALL INHOMOGENEITIES

3.1 Introduction

For the case of a finite crack with tips lodged in vanishingly small inhomogeneities, the asymptotic limit is just the solution for a semi–infinite crack lodged in an inhomogeneity of finite size. The benchmark problem is that of a semi–infinite crack penetrating a circular inhomogeneity which was solved by Stief(1987). Noncircular inhomogeneity cases have been considered by Hutchinson (1986). Dimension–analysis considerations indicate that the asymptotic limit can only depend on the shape of the inhomogeneity, as far as geometric dependence is concerned. This dependence is fully accounted for in our calculation via the use of the readily available Fast Fourier Transform Algorithm. Moreover, the size effect may be obtained by including an additional term in the asymptotic expansion.

A recap of the formulation, together with the introduction of a boundary–layer complex variable ζ, is given in section 3.2. Outer expansion, which is valid away from the crack tips, is presented in section 3.3 in terms of z. Inner expansion of the outer expansion is then used as a guide to establish the inner expansion, which is presented in section 3.4. The application of the Fast Fourier Transform Algorithm to the system of interface boundary conditions is discussed in section 3.5. Some numerical results are presented in section 3.6 mainly for the purpose of illustrating the efficency of the numerical scheme.

3.2 Formulation

Using (2.15) and (2.5), the resultant–free condition along the crack become (c.f. Fig.1.1a):

$$W_1^{\pm}(x;\epsilon) + W_1^{\mp}(x;\epsilon) - f_1^{\mp}(x;\epsilon) = 0 \quad \text{for } |x| \approx 1 \quad , \tag{3.1}$$

$$W_2^{\pm}(x;\epsilon) + W_2^{\mp}(x;\epsilon) - f_2^{\mp}(x;\epsilon) = 0 \quad \text{for } |x| < 1 \quad , \tag{3.2}$$

The resultant–continuity and displacement–continuity conditions along C are :

$$\begin{aligned} & W_2(z_c;\epsilon) + W_2(\bar{z}_c;\epsilon) + (z_c-\bar{z}_c)\overline{W_2'(z_c;\epsilon)} - f_2(\bar{z}_c;\epsilon) \\ & \quad = W_1(z_c;\epsilon) + W_1(\bar{z}_c;\epsilon) + (z_c-\bar{z}_c)\overline{W_1'(z_c;\epsilon)} - f_1(\bar{z}_c;\epsilon), \end{aligned} \tag{3.3}$$

$$\begin{aligned} & W_1(z_c;\epsilon) = \gamma W_2(z_c;\epsilon) \\ & \quad + \gamma^* \left[W_2(z_c;\epsilon) + W_2(\bar{z}_c;\epsilon) + (z_c-\bar{z}_c)\overline{W_2'(z_c;\epsilon)} - f_2(\bar{z}_c;\epsilon) \right], \end{aligned} \tag{3.4}$$

where (3.3) has been used in deducing (3.4). The loading condition at infinity, (2.8), is

$$W_2(z;\epsilon) = Wz, \qquad f_2(z;\epsilon) = fz, \qquad \text{as } z \to \infty. \tag{3.5}$$

As $\epsilon \to 0$ the conditions (3.1),(3.3) and (3.4) disappear altogether . Borrowing the terminology used in thin–airfoil analysis (Van Dyke 1975) we shall call the attending solution the outer expansion. Such an expansion is not valid near $|z| = 1$. Inner expansions near the crack tips must be constructed. For this purpose a boundary layer complex variable ζ is introduced for the neighborhood containing $z = 1$. It is defined by

$$\zeta = \rho\, e^{i\varphi} = \frac{z-1}{\epsilon} \tag{3.6}$$

and the interface C [right] may be conveniently written in the form

$$z_c = 1 + \epsilon\rho_c(\varphi)\, e^{i\varphi}, \quad \zeta = \zeta_c = \rho_c(\varphi)\, e^{i\varphi} \tag{3.7}$$

where the single function $\rho_c(\varphi)$ may be used to define the vanishingly small inhomogeneity (Fig.1.1b). It is assumed that

$$\rho_m \le \rho_c(\varphi) \le \rho_M \qquad \text{and} \qquad \rho_m \le 1 \le \rho_M \tag{3.8}$$

so that ρ_M characterizes the maximum dimension of the inhomogeneity. The quantites ρ_M and ρ_m are the key features of the function $\rho_c(\varphi)$. Finally, the square-root character of a crack tip suggests that the required asymptotic sequence must involve powers of $\epsilon^{1/2}$.

3.3 Outer Expansion ($z \neq \pm 1$)

The outer expansion for W_2 and f_2 are gonvered by (3.2) and (3.5). They are

$$f_2(z;\epsilon) \sim fz + \frac{\sigma}{\sqrt{2}}\epsilon^{1/2}\sum_{n=1}^{\infty}\left\{\frac{A_n(\epsilon)}{\left[\frac{z-1}{\epsilon\rho_m}\right]^n} + \frac{A_n(\epsilon)}{\left[\frac{z+1}{\epsilon\rho_m}\right]^n}\right\} \tag{3.9}$$

$$2W_2(z;\epsilon) - f_2(z;\epsilon) \sim$$

$$\sigma(z^2-1)^{1/2}\left[1 + \sum_{n=1}^{\infty}\left\{\frac{B_n(\epsilon)}{\left[\frac{z-1}{\epsilon\rho_m}\right]^n} + \frac{B_n(\epsilon)}{\left[\frac{z+1}{\epsilon\rho_m}\right]^n}\right\}\right] \tag{3.10}$$

where σ is given by (2.10) and W and f by (2.9), and the coefficients $A_n(\epsilon)$ and $B_n(\epsilon)$ have representations of the forms

$$A_n(\epsilon) = \sum_{m=0}^{\infty} A_{nm}(\epsilon^{1/2})^m \quad , \quad B_n(\epsilon) = \sum_{m=0}^{\infty} B_{nm}(\epsilon^{1/2})^m \,. \tag{3.11}$$

We shall be needing the inner expansions of the outer expansion. These results, after normalized by an as yet undefined factor $\sigma^* / \sqrt{2}$, are

$$
\begin{aligned}
\frac{f_2(1+\epsilon\zeta;\epsilon)}{\sigma^*/\sqrt{2}} \sim\ & \frac{\sqrt{2}f}{\sigma^*} + \epsilon^{1/2}\frac{\sigma}{\sigma^*}\sum_{n=1}^{\infty} A_{n0}(\zeta/\rho_m)^{-n} \\
& + \epsilon\left\{\frac{\sqrt{2}f}{\sigma^*} + \frac{\sigma}{\sigma^*}\sum_{n=1}^{\infty} A_{n1}(\zeta/\rho_m)^{-n}\right\} \\
& + \epsilon^{3/2}\left\{\sum_{n=1}^{\infty} A_{n2}(\zeta/\rho_m)^{-n} + \frac{1}{2}A_{10}\right\}\frac{\sigma}{\sigma^*} + \cdots
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
\frac{2W_2(1+\epsilon\zeta;\epsilon) - f_2(1+\epsilon\zeta;\epsilon)}{\sigma^*/\sqrt{2}} \sim\ & \\
& \epsilon^{1/2}(\zeta \qquad)^{1/2}\left\{1 + \sum_{n=1}^{\infty} B_{n0}(\zeta/\rho_m)^{-n}\right\} \\
& + \epsilon(\zeta \qquad)^{1/2}\left\{\sum_{n=1}^{\infty} B_{n1}(\zeta/\rho_m)^{-n}\right\}\frac{2\sigma}{\sigma^*} \\
& + \epsilon^{3/2}\zeta^{1/2}\left\{\sum_{n=1}^{\infty} B_{n2}(\zeta/\rho_m)^{-n} + \frac{1}{2}B_{10}\right. \\
& \left. + \frac{1}{4}(\zeta \qquad)\left[1 + \sum_{n=1}^{\infty}\frac{B_{n0}}{\zeta^n}\right]\right\}\frac{2\sigma}{\sigma^*} + \cdots
\end{aligned}
\tag{3.13}
$$

The above expansions reduce to the ordinary crack solution as $\epsilon \to 0$ and the Laurent series with unknown coefficients are needed to account for the as yet unknown effects of the vanighingly small inhomogeneities located at $z = \pm 1$. Similar situations have been encountered in crack branching (Wu, 1978) and crack tip contact (Wu,1982). That the Laurent series at $z = \pm 1$ have same coefficients is a result of (2.17). The sigularities are contained in D_1 and have no effects on the analyticity of W_2 and f_2 which are defined in D_2.

3.4 Inner Expansions Near $z = 1$.

For a generic function $F(z;\epsilon)$, the associated inner expansion is defined by

$$F^*(\zeta;\epsilon) \sim F(1+\epsilon\zeta;\epsilon) \Big/ \frac{\sigma^*}{\sqrt{2}} \quad . \tag{3.14}$$

Moreover,

$$F^*(\zeta;\epsilon) \sim \sum_{n=0}^{\infty} (\epsilon^{1/2})^n \, F_n^*(\zeta) \quad . \tag{3.15}$$

It follows that

$$f_2^*(\zeta;\epsilon) \sim \sum_{n=0}^{\infty} (\epsilon^{1/2})^n \, f_{2n}^*(\zeta), \tag{3.16}$$

$$2W_2^*(\zeta;\epsilon) - f_2^*(\zeta;\epsilon) \sim \sum_{n=0}^{\infty} (\epsilon^{1/2})^n \left[2W_{2n}^*(\zeta) - f_{2n}^*(\zeta)\right], \tag{3.17}$$

where the right–hand side terms are just the terms defined in (3.12) and (3.13).

The functions $f_1^*(\zeta;\epsilon)$ and $W_1^*(\zeta;\epsilon)$ must satisfy (3.2) in the appropriately transformed form. This condition is identically satisfied if

$$f_1^*(\zeta;\epsilon) \sim \sum_{n=0}^{\infty} f_{1n}^*(\zeta) \ , \tag{3.18}$$

$$2W_1^*(\zeta;\epsilon) - f_1^*(\zeta;\epsilon) \sim \sum_{n=0}^{\infty} (\epsilon^{1/2})^n \left[2W_{1n}^*(\zeta) - f_{1n}^*(\zeta)\right] , \tag{3.19}$$

and

$$f_{10}^*(\zeta) = \gamma f \Big/ \frac{\sigma^*}{\sqrt{2}} \ , \quad 2W_{10}^*(\zeta) - f_{10}^*(\zeta) = 0 \ .$$

$$f_{1n}^*(\zeta) = \sum_{k=0}^{\infty} a_{k,n-1} (\zeta/\rho_M)^k \ , \tag{3.20}$$

$$2W_{1n}^*(\zeta) - f_{1n}^*(\zeta) = \zeta^{1/2} \sum_{k=0}^{\infty} 2b_{k,n-1} (\zeta/\rho_M)^k \ ,$$

where the last two expressions are for $n \geq 1$. All the unknown coefficients must now be determined to satisfy (3.3) and (3.4) which now become

$$\begin{aligned} &W_{2n}^*(\zeta_c) + W_{2n}^*(\bar{\zeta}_c) + (\zeta_c - \bar{\zeta}_c)\, \overline{W_{2n}^{*\prime}(\zeta_c)} - f_{2n}^*(\bar{\zeta}_c) \\ &\quad = W_{1n}^*(\zeta_c) + W_{1n}^*(\bar{\zeta}_c) + (\zeta_c - \bar{\zeta}_c)\, \overline{W_{1n}^{*\prime}(\zeta_c)} - f_{1n}^*(\bar{\zeta}_c) \ , \end{aligned} \tag{3.21}$$

$$\begin{aligned} W_{1n}^*(\zeta_c) = \gamma W_{2n}^*(\zeta_c) + \gamma^* \Big[&W_{2n}^*(\zeta_c) + W_{2n}^*(\bar{\zeta}_c) \\ &+ (\zeta_c - \bar{\zeta}_c)\, \overline{W_{2n}^{\prime}(\zeta_c)} - f_{2n}^*(\bar{\zeta}_c)\Big] \ . \end{aligned} \tag{3.22}$$

The expansions defined by (3.16) (3.17) (3.12) and (3.13) are valid for the region outside the inhomogeneity, Fig 1.1b, and the unknown coefficients involved in (3.12),(3.13) and (3.20) are determined by applying the standard Fast Fourier Transform (FFT) algorithm to (3.21) and (3.22).

Before implementing the FFT calculation, the normalizing factor σ^* should be properly chosen for specific cases. This is explained in the following section.

3.5 Fast Fourier Transform Algorithm And Numerical Procedure.

It is convenient to consider shear loading σ_{12} and axial loading $(\sigma_{11},\sigma_{22})$ separately. Let us first consider the case $\sigma_{12}=0$. We let $\sigma^*=\sigma_{22}$ and the first 4 sets of unknown sequences become:

$$\left\{\begin{array}{ll} f^*_{20}(\zeta) = \sqrt{2}\, f / \sigma^*, & 2W^*_{20}(\zeta) - f^*_{20}(\zeta)=0 \\ f^*_{10}(\zeta) = \gamma\, f^*_{20}(\zeta), & 2W^*_{10}(\zeta) - f^*_{10}(\zeta)=0 \end{array}\right. \tag{3.23}$$

$$\left\{\begin{array}{l} f^*_{21}(\zeta) = \sum\limits_{n=1}^{\infty} A_{n0}\,(\zeta/\rho_m)^{-n} \\ 2W^*_{21}(\zeta) - f^*_{21}(\zeta) = 2\,(\zeta \qquad)^{1/2} \left\{ 1 + \sum\limits_{n=1}^{\infty} B_{n0}(\zeta/\rho_m)^{-n} \right\} \\ f^*_{11}(\zeta) = \sum\limits_{n=1}^{\infty} a_{n0}(\zeta/\rho_M)^{n} \\ 2W^*_{11}(\zeta) - f^*_{11}(\zeta) = \zeta^{1/2} \sum\limits_{n=0}^{\infty} 2b_{n0}(\zeta/\rho_M)^{n} \end{array}\right. \tag{3.24}$$

$$
\left\{
\begin{aligned}
& f_{22}^*(\zeta) = \sqrt{2}\, f / \sigma^* + \sum_{n=1}^{\infty} A_{n1}(\zeta/\rho_m)^{-n} \\
& 2W_{22}^*(\zeta) - f_{22}^*(\zeta) = 2(\zeta/\rho_m)^{1/2} \sum_{n=1}^{\infty} B_{n1}(\zeta/\rho_m)^{-n} \\
& f_{12}^*(\zeta) = \sum_{n=0}^{\infty} a_{n1}(\zeta/\rho_M)^{n} \\
& 2W_{12}^*(\zeta) - f_{12}^*(\zeta) = \zeta^{1/2} \sum_{n=0}^{\infty} 2b_{n1}(\zeta/\rho_M)^{n}
\end{aligned}
\right.
\tag{3.25}
$$

$$
\left\{
\begin{aligned}
& f_{23}^*(\zeta) = \frac{1}{2} A_{10} + \sum_{n=1}^{\infty} A_{n2}(\zeta/\rho_m)^{-n} \\
& 2W_{23}^*(\zeta) - f_{23}^*(\zeta) = 2(\zeta/\rho_m)^{1/2} \left\{ \frac{1}{2} B_{10} + \sum_{n=1}^{\infty} B_{n2}(\zeta/\rho_m)^{-n} \right. \\
& \qquad \left. + \frac{1}{4}(\zeta/p_m) \left[1 + \sum_{n=1}^{\infty} B_{n0}\, (\zeta/p_m)^{-n} \right] \right\} \\
& f_{13}^*(\zeta) = \sum_{n=0}^{\infty} a_{n2}(\zeta/\rho_M)^{n} \\
& 2W_{13}^*(\zeta) - f_{13}^*(\zeta) = \zeta^{1/2} \sum_{n=0}^{\infty} 2b_{n2}(\zeta/\rho_M)^{n}
\end{aligned}
\right.
\tag{3.26}
$$

where

$$
\frac{\sqrt{2}\, f}{\sigma^*} = \frac{1}{\sqrt{2}} \left[\frac{\sigma_{11}}{\sigma_{22}} - 1 \right]
\tag{3.27}
$$

Each set must satisfy the continuity conditions (3.21) and (3.22). We note that (3.21) is the continuity in resultant force. The first set (3.23) satisfies the conditions identically. For the other sets we truncate the sums at n=N so that a total of 4N+2 unknown coefficients are involved. Since the standard Fast Fourier Transform subroutines FFTCF and FFTCB are written for functions of a real variable, complex functions of the complex variable $\zeta_c = \rho_c(\varphi)e^{i\varphi}$, typified by a generic symbol $G(\zeta_c)$, are represented as follows:

$$G(\zeta_c) = \sum_{n=1}^{N} C_n \zeta_c^n = \sum_{n=1}^{N} \left\{ \sum_{m=0}^{M} C_{mn} e^{im\varphi} + \sum_{m=0}^{-M} C'_{mn} e^{im\varphi} \right\} = G^*(\varphi) \tag{3.28}$$

where M = 500 has been used in all calculations. The truncation number N is determined by a convenient convergence criteria.

We normalize the physical stress intensity factors K_I and K_{II} by $\sigma_{22}\sqrt{\pi}$ and the result is :

$$\begin{aligned} K_1^{(1)} - i\,K_2^{(1)} &= (K_I - i\,K_{II}) \,/\, \sigma_{22}\sqrt{\pi} \\ &\sim b_{00} + \epsilon^{1/2} b_{01} + \epsilon b_{02} + \cdots \end{aligned} \tag{3.29}$$

It is clear from (3.25) and (3.27) that $b_{01} = 0$ if $\sigma_{11} = \sigma_{22}$.

For a pure shear loading condition, $\sigma_{11} = \sigma_{22} = 0$ and we choose $\sigma^* = -i\sigma_{12}$. Equations (3.23) – (3.26) remain unchanged and (3.27) is replaced by

$$\frac{\sqrt{2}f}{\sigma^*} = -\sqrt{2} \tag{3.30}$$

The associated stress intensity factors are given by

$$K_2^{(2)} + i\, K_1^{(2)} = (K_{II} + i\, K_I) \,/\, \sigma_{12}\sqrt{\pi}$$

$$\sim b_{00} + \epsilon^{1/2} b_{01} + \epsilon b_{02} + \cdots\cdots \tag{3.31}$$

It is clear from the solution procedure that the solutions for (3.24) and (3.26) are identical in form for both loading conditions. The solution for (3.25) is determined by either (3.27) or (3.30) Thus, b_{01} is never zero in (3.31). On the other hand, b_{00} and b_{02} for both cases are identical. The actual field variables, however, must be modified by the factor σ^* and hence are completely different for the two cases.

3.6 Results and Discussion

As we have mentioned in the introduction that the $\epsilon \to 0$ limit depends only on the shape of the inhomogeneity, as far as geometric dependence is concerned. To illustrate this dependence and also to test the efficiency of our computer code, the class of problems depicted in Fig. 3.1 is considered. The asymptotic configuration is that of a semi–infinite crack with a tip inhomogeneity of thickness 2 and length 2 + a+ b. We mention in passing that the actural physical dimensions are $2\epsilon \times (2 + a + b)\, \epsilon$. The material properties are fixed by $\nu_1 = \nu_2 = 0.2$ and $\mu_1/\mu_2 = 0.5$ so that the parametric study is purely geomotrical. Moreover, only mode–I loading is considered in the illustration.

The benchmark situation of a circular inhomogeneity is recovered by setting a = b = 0 and the associated normalized SIF is 0.645 (Steif 1987). It is anticipated that increasing a would lead to an decrease in SIF while increasing b would actually intensify the associated SIF (Hutchinson 1986). Every method has its limitations and the present code cannot be expected to function for cases where a and b are very much greater than 1.

Considered as a function of a and b, the normalized SIF, K (a, b), is expected to approach to certain asymptotic limits very rapidly. K (∞, b) is just the solution for problem II and K (∞, ∞) $\approx$ 0.74 which is extrapolated from the results given in (Hilton and Sih, 1970).

The pertinent results are protted in Fig. 3.2 as a function of b with a as a parameter. The a = ∞ curve is produced by our second computer code which will be developed in Chapter IV. It is seen that the results conform with all the expected trends.

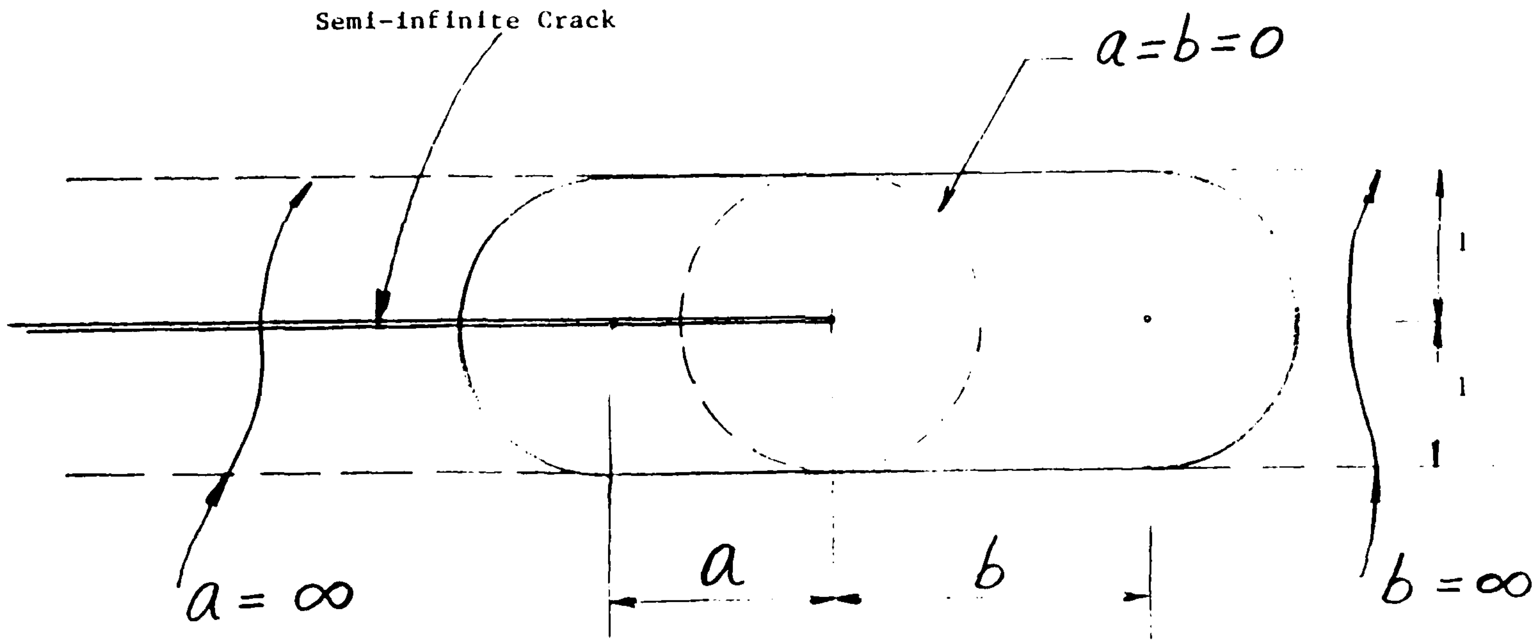


Fig. 3.1 Configuration of a crack with a tip inhomogeneity of thickness 2 and length $2\epsilon \cdot (2+a+b)$.

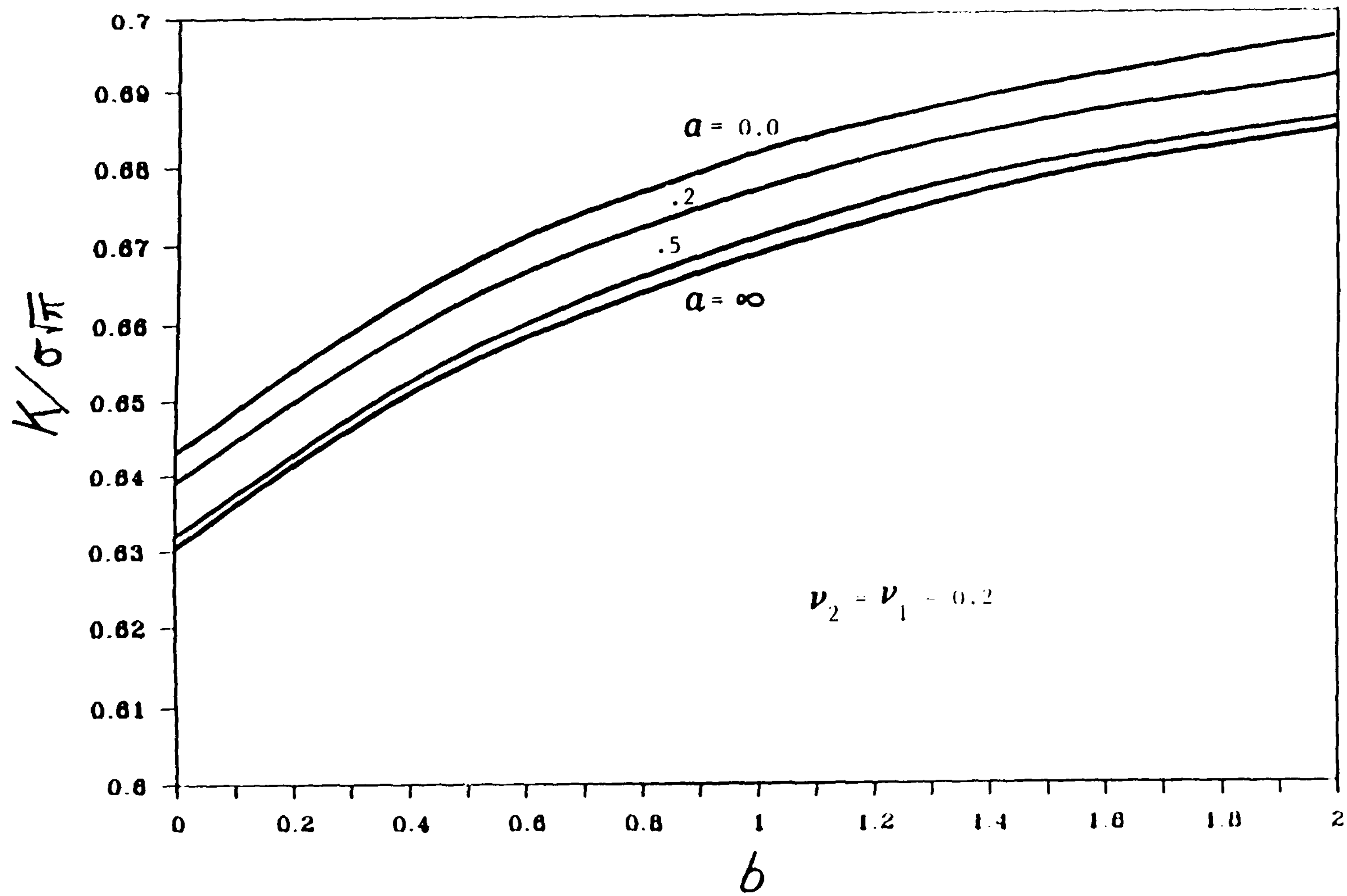


Fig. 3.2 K_1 as a function of b with a as a parameter

CHAPTER IV

A CRACK IN A THIN INHOMOGENEITY

4.1 Introduction

The present problem differs from that of Chapter III in that the complete crack is wholly surrounded by the inhomogeneity. As a result, the infinite medium contains no portion of the traction–free boundary and hence the analytical structure of the solution for the medium in the vicinity of a crack tip is not known. For this reason and a' , the purpose of setting up a benchmark for validation, the problem of a crack imbeded inside of a confocal elliptic inhomogeneity is first considered in detail.

A Fourier series solution is first constructed in section 4.2. The analysis is preceeded by a brief discussion of the associated antiplane shear problem for which the exact solution is presented. The exact asymptotic limits for very large and very small inhomogeneities are deduced from the exact solution. Their exact dependence on size and moduli serves as a guidence for the desired plane problem solution.

The Fourier series solution obtained for the plane problem is considered to be numerically exact, as the convergence can be easily checked by varying the number of terms included in the actual computation. The convergence of the series, however, becomes extremely slow as the inhomogeneity becomes vanishingly thin. The asymptotic limit for the SIF, though, is easily extrapolated from the numerical results.

The series solution sheds no light on the desired analytical structure of the solution. A complete asymptotic analysis is therefore carried out in section 4.3. The needed analytical structure is revealed by the inner expansion of the outer expansion. As an unexpected useful byproduct, an approximate but explicit solution is obtained for the confocal situation. This completes the establishment of the benchmark solution.

The general case is finally presented in section 4.4 where the thickness of the inhomogeneity and the nose thickness, i.e., the length of the inhomogeneity minus the crack length, are assumed to have the same order of magnitude ϵ. We note that for the confocal situation the nose thickness is of the order of ϵ^2 while the inhomogeneity thickness is of the order ϵ. In this regard and in the context of the final boundary layer computation, the confocal situation is even more difficult than our general case, as the former requires the satisfying of boundary conditions specified on a parabola in the boundary–layer variable. Neverthless, the analytic structure revealed by the benchmark analysis is the key to the success of the general representation. The unknown constents involved in the series solution are again determined by the Fast Fourier Transform Algorithm.

4.2 Series Solution

4.2.1 Introduction

A family of confocal ellipses may be characterized by a single parameter $\rho > 1$. In the limit as $\rho \to 1$ the ellipse degenerates into a straight line of length 2. It tends to a circle of infinite radius as $\rho \to \infty$. The geometry of the problem is fixed by the crack ($\rho = 1$) and the size of an elliptic inhomogeneity ($\rho = \rho_0$). Both the inhomogeneity and the infinite medium are assumed to be homogeneous and isotropic.

Section 4.2.2 summarizes the formulation in a transformed complex plane. The exact solution for the anti-plane shear case is presented in Section 4.2.3. Explicit asymptotic limits for large and small inhomogeneities are extracted from the exact formula. Plane problems are dealt with in Section 4.2.4. The case of equal shear modulus and unequal Poisson's ratio is solved exactly, and the general case is presented as a series solution.

A large number of references on inhomogeneity problems may be found in Mura (1982,1988), but we have not found any reference dealing with the consideration of a crack in a vanishingly small inhomogeneity. The closest situation is the one given by Warren (1983), who considered the edge dislocation inside an elliptical inhomogeneity, including vanishingly small inhomogeneities.

Numerical results for the plane problems are presented only for plane strain and Mode-I conditions. Parametric dependence of SIF on the size of the inhomogeneity is discussed in detail in Section 4.2.5 for the range where the inhomogeneity is softer than the medium.

4.2.2 Formulation

Let (z_1,z_2) be rectangular Cartesian coordinates and $z = z_1 + iz_2$ the associated complex variable in the z–plane. A crack of length 2 in the z–plane is mapped onto a unit circle in a new complex ζ–plane via the mapping function

$$z = m(\zeta) = \frac{1}{2}\left[\zeta + \frac{1}{\zeta}\right] \tag{4.2.1}$$

where $\zeta = \zeta_1 + i\zeta_2 = \rho e^{i\theta}$ (Fig. 4.2.1b) and $\rho > 1$. The image of the circle $\zeta = \zeta_o = \rho_o e^{i\theta}$ is the ellipse (Fig. 4.2.1a)

$$(z_1/a)^2 + (z_2/b)^2 = 1 \tag{4.2.2}$$

where

$$a = \frac{1}{2}\left[\rho_o + \frac{1}{\rho_o}\right], \quad b = \frac{1}{2}\left[\rho_o - \frac{1}{\rho_o}\right]. \tag{4.2.3}$$

The infinite z–plane is now conveniently divided into two regions D_1 and D_2 by the single parameter ρ_o, viz.,

$$D_1:\ 1 < \rho < \rho_o, \quad D_2:\ \rho > \rho_o. \tag{4.2.4}$$

We shall call D_1 the inhomogeneity and D_2 the medium. The two regions are of different elastic materials characterized by shear moduli μ_a and plane–elasticity constants

$$\kappa_a = \begin{cases} 3-4\nu_a & \text{plane strain} \\ (3-\nu_a)/(1+\nu_a) & \text{plane stress} \end{cases} \tag{4.2.5}$$

where ν_a are Poisson's ratios. The infinite plane is loaded at infinity by

$$\text{Anti–plane:}\quad \tau_{3a} = \sigma_{3a} \quad \text{as } |z| \to \infty, \tag{4.2.6}$$

$$\text{Plane:}\quad \tau_{a\beta} = \sigma_{a\beta} \quad \text{as } |z| \to \infty, \tag{4.2.7}$$

where τ_{ij} are the stress components.

For the anti-plane problem, the displacement $u_3(z_1,z_2)$, stresses $\tau_{3\alpha}(z_1,z_2)$ and resultant force R_3 along an arc may be expressed in terms of a single complex function $F(z)$. We have

$$u_3 = \frac{1}{2}\left[\Phi(\zeta) + \overline{\Phi(\zeta)}\right], \tag{4.2.8}$$

$$\tau_{31} - i\tau_{32} = \mu\Phi'(\zeta)/m'(\zeta), \tag{4.2.9}$$

$$R_3 = -i\frac{\mu}{2}\left[\Phi(\zeta) + \overline{\Phi(\zeta)}\right], \tag{4.2.10}$$

where $\Phi(\zeta) = F(m(\zeta))$ and $m(\zeta)$ is the mapping function (4.2.1).

For the plane problem, the displacements $u_\alpha(z_1,z_2)$, stresses $\tau_{\alpha\beta}(z_1,z_2)$ and resultant force $R_1 + iR_2 = R$ along an arc may be expressed in terms of two complex functions $W(z)$ and $w(z)$. We have

$$2\mu(u_1 + iu_2) = \kappa\Omega(\zeta) - \frac{m(\zeta)}{\overline{m'(\zeta)}}\overline{\Omega'(\zeta)} - \overline{\omega(\zeta)}, \tag{4.2.11}$$

$$iR = \Omega(\zeta) + \frac{m(\zeta)}{\overline{m'(\zeta)}}\overline{\Omega'(\zeta)} - \overline{\omega(\zeta)}, \tag{4.2.12}$$

where $\Omega(\zeta) = W(m(\zeta))$ and $\omega(\zeta) = w(m(\zeta))$.

The complex functions must be determined for the two regions D_1 and D_2 subjected to the loading conditions, (4.2.6) and (4.2.7), and the continuity conditions along the interface boundary characterized by ρ_o. The crack surface is assumed to be traction free. The traction–free condition may be integrated along the crack to become a resultant–free condition. Similarly, traction continuity along the interface may be integrated to become a resultant continuity condition. The integrated forms of these conditions will be used in the calculations to follow.

We shall place a subscript α on a complex function to indicate its region of definition. For example, $F_\alpha(z)$ and $\Phi_\alpha(\zeta)$ are defined for region D_α.

4.2.3 Anti–Plane Shear

The problem may be most conveniently solved in the ζ–plane. The integrated traction–free, integrated traction continuity, displacement continuity and loading conditions are

$$\Phi_1(e^{i\theta}) - \overline{\Phi_1(e^{i\theta})} = 0 \,, \tag{4.2.13}$$

$$\mu_1\left[\Phi_1(\zeta_o) - \overline{\Phi_1(\zeta_o)}\right] = \mu_2\left[\Phi_2(\zeta_o) - \overline{\Phi_2(\zeta_o)}\right] , \tag{4.2.14}$$

$$\Phi_1(\zeta_o) + \overline{\Phi_1(\zeta_o)} = \Phi_2(\zeta_o) + \overline{\Phi_2(\zeta_o)} \,, \tag{4.2.15}$$

$$\Phi_2 = \Phi\zeta \quad \text{as } \zeta \to \infty \,, \tag{4.2.16}$$

where $\zeta_o = \rho_o e^{i\theta}$ and

$$\Phi = \frac{1}{2}\frac{1}{\mu_2}(\sigma_{31} - i\sigma_{32}) \,. \tag{4.2.17}$$

The solution is

$$\Phi_1(\zeta) = A\zeta + \frac{\bar{A}}{\zeta} \qquad (1 < |\zeta| < \rho_o) \,, \tag{4.2.18}$$

$$\Phi_2(\zeta) = \Phi\zeta + \left[(1+\rho_o^2)\bar{A} - \rho_o^2\bar{\Phi}\right]\frac{1}{\zeta} \qquad (|\zeta| > \rho_o) \,, \tag{4.2.19}$$

where

$$A = 2\rho_o^2\Phi \Big/ \left[(\rho_o^2+1) + \frac{\mu_1}{\mu_2}(\rho_o^2-1)\right] . \tag{4.2.20}$$

The stress intensity factor may be readily determined. It is convenient to normalize the SIF K_{III} by the factor $\sigma_{32}\sqrt{\pi}$, and the result is

$$K_3\left[\rho_o, \frac{\mu_1}{\mu_2}\right] = \frac{K_{III}}{\sigma_{32}\sqrt{\pi}} = \frac{\mu_1}{\mu_2}\left[\frac{2\rho_o^2}{\rho_o^2+1}\frac{1}{1+\frac{\mu_1}{\mu_2}\frac{\rho_o^2-1}{\rho_o^2+1}}\right]. \tag{4.2.21}$$

The following limits may be easily obtained

$$K_3\left[\infty, \frac{\mu_1}{\mu_2}\right] = 2\frac{\mu_1}{\mu_2} \Big/ \left[1, \frac{\mu_1}{\mu_2}\right], \tag{4.2.22}$$

$$K_3\left[1, \frac{\mu_1}{\mu_2}\right] = \frac{\mu_1}{\mu_2}. \tag{4.2.23}$$

These two limits are plotted in Fig. 4.2.2. The sign of $\partial K_3/\partial\rho_o$ is governed by $(1 - \mu_1/\mu_2)$. Thus

$$K_3\left[1, \frac{\mu_1}{\mu_2}\right] \lesseqqgtr K_3\left[\rho_o, \frac{\mu_1}{\mu_2}\right] \lesseqqgtr K_3\left[\infty, \frac{\mu_1}{\mu_2}\right] \quad \text{if } \frac{\mu_1}{\mu_2} \lesseqqgtr 1. \tag{4.2.24}$$

It is noted that the bounds are of practical significance for the cases where the inhomogeneity is softer than the medium.

A very slender inhomogeneity may be defined by $\rho_o = 1+\epsilon$ where $\epsilon << 1$. The exact result (4.2.21) may be used to obtain

$$K_3\left[1+\epsilon, \frac{\mu_1}{\mu_2}\right] \sim \frac{\mu_1}{\mu_2}\left[1+\epsilon\left[1-\frac{\mu_1}{\mu_2}\right] + \cdots\right] \tag{4.2.25}$$

provided that $\epsilon(\mu_1/\mu_2) \to 0$ as $\epsilon \to 0$. It is, therefore, clear that such a two-term asymptotic expansion is valid only for the cases where the inhomogeneity is either softer or slightly harder than the matrix.

4.2.4 Plane Problem

Since the plane problems cannot be solved exactly, we begin by constructing a series solution in the ζ-plane. The traction-free condition on the unit circle enables us to introduce the stress continuation (England, 1971)

$$\omega_1(\zeta) = \overline{-\Omega_1(1/\bar{\zeta})} - \frac{\overline{m(1/\bar{\zeta})}}{m'(\zeta)}\,\Omega'_1(\zeta)\,. \tag{4.2.26}$$

The function $\Omega_1(\zeta)$ is now extended to the region $1/\rho_0 < \rho < \rho_0$, and the traction-free condition on $\rho = 1$ is identically satisfied. The conditions (4.2.7) are met if

$$\Omega_2(\zeta) = \Omega\zeta + O\left[\frac{1}{\zeta}\right] \qquad |\zeta| \to \infty\,, \tag{4.2.27}$$

$$\omega_2(\zeta) = \omega\zeta + O\left[\frac{1}{\zeta}\right] \qquad |\zeta| \to \infty\,, \tag{4.2.28}$$

where

$$\Omega = \frac{1}{8}\left(\sigma_{11} + \sigma_{22}\right)\,, \quad \omega = \frac{1}{4}\left[(\sigma_{22} - \sigma_{11}) + i2\sigma_{12}\right]\,. \tag{4.2.29}$$

The integrated form of the traction continuity condition along ζ_0 may be obtained from (4.2.12), i.e.,

$$\Omega_2(\zeta_0) + \frac{m(\zeta_0)}{\overline{m'(\zeta_0)}}\,\overline{\Omega'_2(\zeta_0)} + \overline{\omega_2(\zeta_0)}$$

$$= \Omega_1(\zeta_0) - \Omega_1\left[\frac{1}{\bar{\zeta}_0}\right] + \left[m(\zeta_0) - m\left[\frac{1}{\bar{\zeta}_0}\right]\right]\frac{\overline{\Omega'_1(\zeta_0)}}{\overline{m'(\zeta_0)}} \tag{4.2.30}$$

where (4.2.26) has been applied. Continuity in displacements along ζ_0 yields

$$\frac{1}{\mu_2}\left\{\kappa_2\Omega_2(\zeta_o) - \frac{m(\zeta_o)}{\overline{m'(\zeta_o)}}\,\overline{\Omega_2'(\zeta_o)} - \overline{\omega_2(\zeta_o)}\right\}$$

$$= \frac{1}{\mu_1}\left\{\kappa_1\Omega_1(\zeta_o) + \Omega_1\left[\frac{1}{\zeta_o}\right] - \left[m(\zeta_o) - m\left[\frac{1}{\zeta_o}\right]\right]\frac{\overline{\Omega_1'(\zeta_o)}}{\overline{m'(\zeta_o)}}\right\} \tag{4.2.31}$$

which, after applying (4.2.30), may be reduced to

$$\Omega_1(\zeta_o) = \gamma\Omega_2(\zeta_o) + \gamma^*\,\Omega_2(\zeta_o) + \left\{\frac{m(\zeta_o)}{\overline{m'(\zeta_o)}}\,\overline{\Omega_2'(\zeta_o)} - \overline{\omega_2(\zeta_o)}\right\} \tag{4.2.32}$$

where

$$\gamma = \frac{(1+\kappa_2)\mu_1}{(1+\kappa_1)\mu_2}, \qquad \gamma^* = \frac{1}{1+\kappa_1}\left[1 - \frac{\mu_1}{\mu_2}\right] \tag{4.2.33}$$

are two composite parameters (Dundurs, 1969). We note that (4.2.30) may be deduced from (4.2.31) via the relation

$$(4.5) \equiv \left[\text{Letting } \mu_1 = \mu_2 = 1 \text{ and } \kappa_1 = \kappa_2 = -1 \text{ in (4.2.31)}\right] \tag{4.2.34}$$

Before proceeding, we shall first consider the special case $\mu_1 = \mu_2$.

i) **Exact Solution for Equal Shear Modulus**

For this case, the composite parameters defined by (4.2.33) become

$$\gamma = \gamma_o = \frac{1+\kappa_2}{1+\kappa_1}, \qquad \gamma^* = 0 \tag{4.2.35}$$

and (4.2.32) becomes

$$\Omega_1(\zeta_o) = \gamma_o \Omega_2(\zeta_o), \tag{4.2.36}$$

which serves as an analytic continuation of the two functions $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$. Making the substitution $\Omega_1(\zeta) = \gamma_o\Omega_2(\zeta)$, we obtain from (4.2.30)

$$\overline{\omega_2(\rho_o^2/\bar{\zeta}_o)} + \frac{m(\zeta_o)}{\overline{m'(\rho_o^2/\bar{\zeta}_o)}}\overline{\Omega_2'(\rho_o^2/\bar{\zeta}_o)} - \gamma_o M(\zeta_o)\overline{\Omega_2'(\rho_o^2/\bar{\zeta}_o)}$$

$$= (\gamma_o-1)\Omega_2(\zeta_o) - \gamma_o\Omega_2(\zeta_o/\rho_o^2) \tag{4.2.37}$$

where

$$M(\zeta_o) = \frac{m(\zeta_o) - m1/\bar{\zeta}_o)}{\overline{m'(\zeta_o)}} = -\frac{\rho_o^2(\rho_o^2-1)(\zeta_o^2-\rho_o^2)}{\zeta_o(\zeta_o^2-\rho_o^4)}. \tag{4.2.38}$$

It follows from (4.2.37) that the function $H(\zeta)$ defined by

$$H(\zeta) = \begin{cases} (\gamma_o-1)\Omega_2(\zeta) - \gamma_o\Omega_2(\zeta/\rho_o^2), & (|\zeta| > \rho_o) \\ \overline{\omega_2(\rho_o^2/\bar{\zeta}_o)} + \dfrac{m(\zeta)}{\overline{m'(\rho_o^2/\bar{\zeta}_o)}}\overline{\Omega_2'(\rho_o^2/\bar{\zeta}_o)} - \gamma_o M(\zeta)\overline{\Omega_2'(\rho_o^2/\bar{\zeta}_o)}, & (|\zeta| < \rho_o) \end{cases} \tag{4.2.39}$$

is holomorphic in the whole ζ-plane. Moreover, its properties at $\zeta = 0$ and ∞ are governed by the right-hand side of (4.2.39). The complete solution is

$$H(\zeta) = \left[-1 + \gamma_o\left[1 - \frac{1}{\rho_o^2}\right]\right]\Omega\zeta + \left\{\bar{\omega}\rho_o^2 + [1+\gamma_o(\rho_o^2-1)]\Omega\right\}\frac{1}{\zeta}, \tag{4.2.40}$$

$$\Omega_2(\zeta) = \Omega\zeta - \left[\Omega + \frac{\bar{\omega}\rho_o^2}{1 + \gamma_o(\rho_o^2-1)}\right]\frac{1}{\zeta}, \tag{4.2.41}$$

$$\omega_2(\zeta) = \overline{H(\rho_o^2/\bar{\zeta})} + \gamma_o\overline{M(\rho_o^2/\bar{\zeta})}\Omega_2'(\zeta) - \left[\overline{m(\rho_o^2/\bar{\zeta})}/m'(\zeta)\right]\Omega_2'(\zeta), \tag{4.2.42}$$

$$\Omega_1(\zeta) = \gamma_o\Omega_2(\zeta). \tag{4.2.43}$$

Let us use the factors $\sigma_{22}\sqrt{\pi}$ and $\sigma_{12}\sqrt{\pi}$ to normalize the SIF's K_I and K_{II}, and write

$$K_1 = K_I/\sigma_{22}\sqrt{\pi}, \quad K_2 = K_{II}/\sigma_{12}\sqrt{\pi}. \tag{4.2.44}$$

The following explicit results are readily obtained

$$K_1 = \frac{\gamma_o[(\rho_o^2+1) + \gamma_o(\rho_o^2-1)]}{2[1 + \gamma_o(\rho_o^2-1)]} - \frac{\gamma_o(1-\gamma_o)(\rho_o^2-1)}{2[1 + \gamma_o(\rho_o^2-1)]}\frac{\sigma_{11}}{\sigma_{22}}, \tag{4.2.45}$$

$$K_2 = \frac{\gamma_o\rho_o^2}{1 + \gamma_o(\rho_o^2-1)}. \tag{4.2.46}$$

There are the following exact limits

$$\lim_{\rho_o\to 1} K_1 \quad \text{and} \quad K_2 = \gamma_o = \frac{1+\kappa_2}{1+\kappa_1}, \tag{4.2.47}$$

$$\lim_{\rho_o\to\infty} K_1 = \frac{1}{2}(1+\gamma_o) - \frac{1}{2}(1-\gamma_o)\frac{\sigma_{11}}{\sigma_{22}}, \tag{4.2.48}$$

$$\lim_{\rho_o\to\infty} K_2 = 1. \tag{4.2.49}$$

ii) <u>Series Solution</u>

The complex functions Ω_1, Ω_2 and ω_2, together with their regions of definition, admit the following series representations

$$\Omega_1(\zeta) = \sum_{n=1}^{\infty} \rho_o^{1-n}\left[A_n\zeta^n + a_n\zeta^{-n}\right], \tag{4.2.50}$$

$$\Omega_2(\zeta) = \Omega\zeta + \sum_{n=1}^{\infty} \rho_o^{1+n}B_n\zeta^{-n}, \tag{4.2.51}$$

$$\omega_2(\zeta) = \omega\zeta + \sum_{n=1}^{\infty} \rho_o^{1+n}b_n\zeta^{-n}, \tag{4.2.52}$$

where $n = 1,3,5,\ldots$ The factors ρ_o^{1-n} and ρ_o^{1+n} are included for convenience and pose no restrictions on the validity of the series representations. They are, nevertheless, conceived from the fact that u_a along $|\zeta| = \rho_o$ must be of the order of ρ_o as $\rho_o \to \infty$.

Substituting the above into (4.2.31), setting $\zeta = \rho_o e^{i\theta}$, and equating coefficients of $e^{in\theta}$ to zero, we obtain

$$\begin{aligned}
&-n\left[1 - \frac{1}{\rho_o^2}\right]\bar{A}_n + (n+2)\left[1 - \frac{1}{\rho_o^2}\right]\bar{A}_{n+2} - \left[1 + \frac{\kappa_1}{\rho_o^{2n}}\right]a_n \\
&\quad + \frac{1}{\rho_o^2}\left[1 + \frac{\kappa_1}{\rho_o^{2(n+2)}}\right]a_{n+2} + \frac{\mu_1}{\mu_2}\kappa_2\left[B_n - \frac{1}{\rho_o^2}B_{n+2}\right] \\
&\quad = \begin{cases} \dfrac{\mu_1}{\mu_2}\bar{\omega} + \dfrac{1}{\rho_o^2} & \text{for } n = 1 \\[2ex] 0 & \text{for } n = 3,5,7,\ldots \end{cases}
\end{aligned} \tag{4.2.53}$$

$$-\left[\kappa_1 + \frac{1}{\rho_o^2}\right]A_1 + \left[1 - \frac{1}{\rho_o^2}\right]\bar{A}_1 + \frac{1}{\rho_o^2}\left[1 - \frac{1}{\rho_o^2}\right]\bar{a}_1 + \frac{1}{\rho_o^2}\left[1 - \frac{\kappa_1}{\rho_o^2}\right]a_1$$

$$+ \frac{\mu_1}{\mu_2}\left[\frac{1}{\rho_o^2}\bar{B}_1 - \bar{b}_1 - \frac{\kappa_2}{\rho_o^2}B_1\right] = -\frac{\mu_1}{\mu_2}\left[(\kappa_2 - 1)\Omega + \frac{1}{\rho_o^2}\bar{\omega}\right], \tag{4.2.54}$$

and

$$-\left[\kappa_1 - \frac{1}{\rho_o^{2n}}\right]\bar{A}_2 + \frac{1}{\rho_o^2}\left[\kappa_1 + \frac{1}{\rho_o^{2(n-2)}}\right]A_{n-2} + \frac{n}{\rho_o^{2n}}\left[1 - \frac{1}{\rho_o^2}\right]\bar{a}_n$$

$$-\frac{n-2}{\rho_o^{2(n-2)}}\left[1 - \frac{1}{\rho_o^2}\right]\bar{a}_{n-2} + \frac{\mu_1}{\mu_2}\left[(n-2)\bar{B}_{n-2} + \frac{n}{\rho_o^2}\bar{B}_n - \bar{b}_n + \frac{1}{\rho_o^2}\bar{b}_{n-2}\right]$$

$$= \begin{cases} \dfrac{\mu_1}{\mu_2}\kappa_2 + \dfrac{\Omega}{\rho_o^2} & \text{for } n = 3 \\ \\ 0 & \text{for } n = 5,7,9,\ldots \end{cases} . \tag{4.2.55}$$

The relations derived from (4.2.30) are obtained by the substitution (4.2.34), i.e.,

$$[\text{Letting } \mu_1 = \mu_2 = 1 \text{ and } \kappa_1 = \kappa_2 = -1 \text{ in } (4.2.53),(4.2.54),(4.2.55)] . \tag{4.2.56}$$

The infinite system (4.2.53)–(4.2.56) are truncated and the resulting finite system is inverted numerically. The SIF's

$$K_I - iK_{II} = 2(\pi)^{1/2}\Omega_1'(1) \tag{4.2.57}$$

are then normalized in accordance with (4.2.44). Values of plane–strain K_1 are plotted in Fig. 4.2.3 for the case $\nu_1 = \nu_2 = 0.2$ and $\sigma_{11} = 0$. The numerical results approach to a limit very rapidly as $\rho_o \to \infty$. The analytic expression for this limit is determined in the next sub–section.

The convergence of the numerical scheme becomes extremely slow as $\rho_o \to 1$. For this reason, values of K_1 for fixed values of μ_1/μ_2 are plotted as functions of $1/\rho_o^2$ in Fig. 4.2.4. It is seen that all curves tend to finite limits as $\rho_o \to 1$. The $\rho_o = 1$ curve indicated in Fig. 4.2.3 is extrapolated from Fig. 4.2.4.

iii) Very Large Inhomogeneity

As we have indicated before that the factors $\rho_o^{1 \pm n}$ built in (4.2.50)–(4.2.52) are conceived from the fact that the displacements u_a along $|\zeta| = \rho_o$ must be of the order of ρ_o as $\rho_o \to \infty$. Thus the series representation may also be interpreted as an asymptotic expansion for large ρ_o. In this interpretation, however, the four sets of constants must be re–expanded in powers of ρ_o^{-2}, i.e.,

$$(A,a,B,b)_n = (\;)_{no} + \frac{1}{\rho_o^2}(\;)_{n1} + \cdots \tag{4.2.58}$$

Substituting the above into (4.2.53)–(4.2.56) and equating to zero the coefficients of powers of ρ_o^{-2}, we obtain an infinite system of infinitely many equations for the determination of the coefficients $(\;)_{nm}$. The asymptotic limit for the case of a very large inhomogeneity is thus governed by the coefficients $(\;)_{no}$.

The system governing $(\;)_{no}$ actually decouples into finite systems and the first cluster of equations are

$$-\bar{A}_{10} + 3\bar{A}_{30} - a_{10} + \frac{\mu_1}{\mu_2}\kappa_2 B_{10} = \frac{\mu_1}{\mu_2}\bar{\omega}\,, \tag{4.2.59}$$

$$-\bar{A}_{10} + 3\bar{A}_{30} - a_{10} - B_{10} = \bar{\omega}\,, \tag{4.2.60}$$

$$-\kappa_1 A_{10} + \bar{A}_{10} - \frac{\mu_1}{\mu_2}\bar{b}_{10} = -\frac{\mu_1}{\mu_2}(\kappa_2 - 1)\Omega\,, \tag{4.2.61}$$

$$A_{10} + \bar{A}_{10} - \bar{b}_{10} = 2\Omega\,, \tag{4.2.62}$$

$$-\kappa_1 A_{30} + \frac{\mu_1}{\mu_2}(\bar{B}_{10} - \bar{b}_{30}) = 0\ , \tag{4.2.63}$$

$$A_{30} + \bar{B}_{10} - \bar{b}_{30} = 0\ , \tag{4.2.64}$$

which may be explicitly solved to yield

$$A_{10} = \frac{\mu_1}{\mu_2}(\kappa_2+1)\Omega/\left[\kappa_1 - 1- + 2\frac{\mu_1}{\mu_2}\right]\ ,$$

$$B_{10} = b_{30} = \left[\frac{\mu_1}{\mu_2} - 1\right]\bar{\omega}/+\left[\frac{\mu_1}{\mu_2}\kappa_2 + 1\right]\ , \tag{4.2.65}$$

$$A_{10} = -B_{10} - A_{10} - \bar{\omega}$$

$$A_{30} = 0\ , \qquad b_{10} = 2(A_{10} - \Omega)\ .$$

In fact, the second cluster of equations yields

$$A_{50} = a_{30} = B_{30} = b_{50} = 0\ . \tag{4.2.66}$$

The asymptotic limit for Ω_1 is merely

$$\Omega_1(\zeta) \sim \left[A_{10}\zeta + a_{10}\frac{1}{\zeta}\right] + O\left[\frac{1}{\rho_o^2}\right] \tag{4.2.67}$$

and

$$\Omega_1'(1) \sim (A_{10} - a_{10}) + O\left[\frac{1}{\rho_o^2}\right] \tag{4.2.68}$$

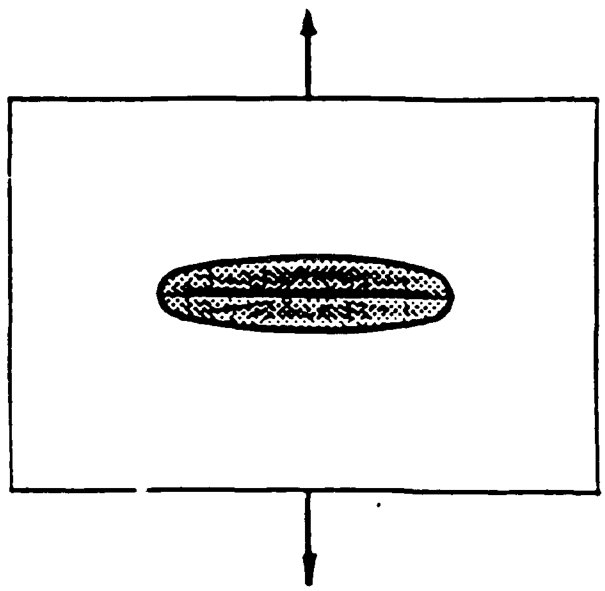
Equations (4.2.44), (4.2.57) and the above lead to the following explicit asymptotic limits for very large elliptic inhomogeneity:

$$K_1 \sim \frac{1}{2}\frac{\mu_1}{\mu_2}(1+\kappa_2)\left[\frac{1-\dfrac{\sigma_{11}}{\sigma_{12}}}{1+\dfrac{\mu_1}{\mu_2}\kappa_2} - \frac{1+\dfrac{\sigma_{11}}{\sigma_{22}}}{\kappa_1 - 1 + 2\dfrac{\mu_1}{\mu_2}}\right], \tag{4.2.69}$$

$$K_2 \sim \frac{\mu_1}{\mu_2}(1+\kappa_2)\Big/\left[1+\frac{\mu_1}{\mu_2}\kappa_2\right]. \tag{4.2.70}$$

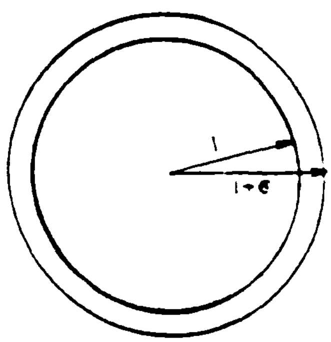
Equation (4.2.69) is in perfect agreement with the numerical asymptotic limit given in Fig. 4.2.3.

The series solution provides us with a complete family of numerically exact benchmark results. Still, the results yield no useful analytic information concerning the behavior of the solution for the medium in the vicinity of the crack tips. This important information will be deduced from the asymptotic analysis of section 4.3.



$$(z_1/a)^2 - (z_2/b)^2 = 1$$

(a)



$$z = m(\zeta) = \frac{1}{2}\left(\zeta - \frac{1}{\zeta}\right)$$

$$\zeta = \zeta_1 + i\zeta_2 = \rho e^{i\sigma}$$

$$\rho = 1. + \epsilon$$

(b)

Fig. 4.2.1 A crack in a confocall elliptic inhomogeneity
a) Configuration in the physical z–plane
b) The image of of the circle $\zeta_o = \rho_o e^{i\theta}$ is the ellipse and the unit circle is the crack in the new complex ζ–plane via mapping function

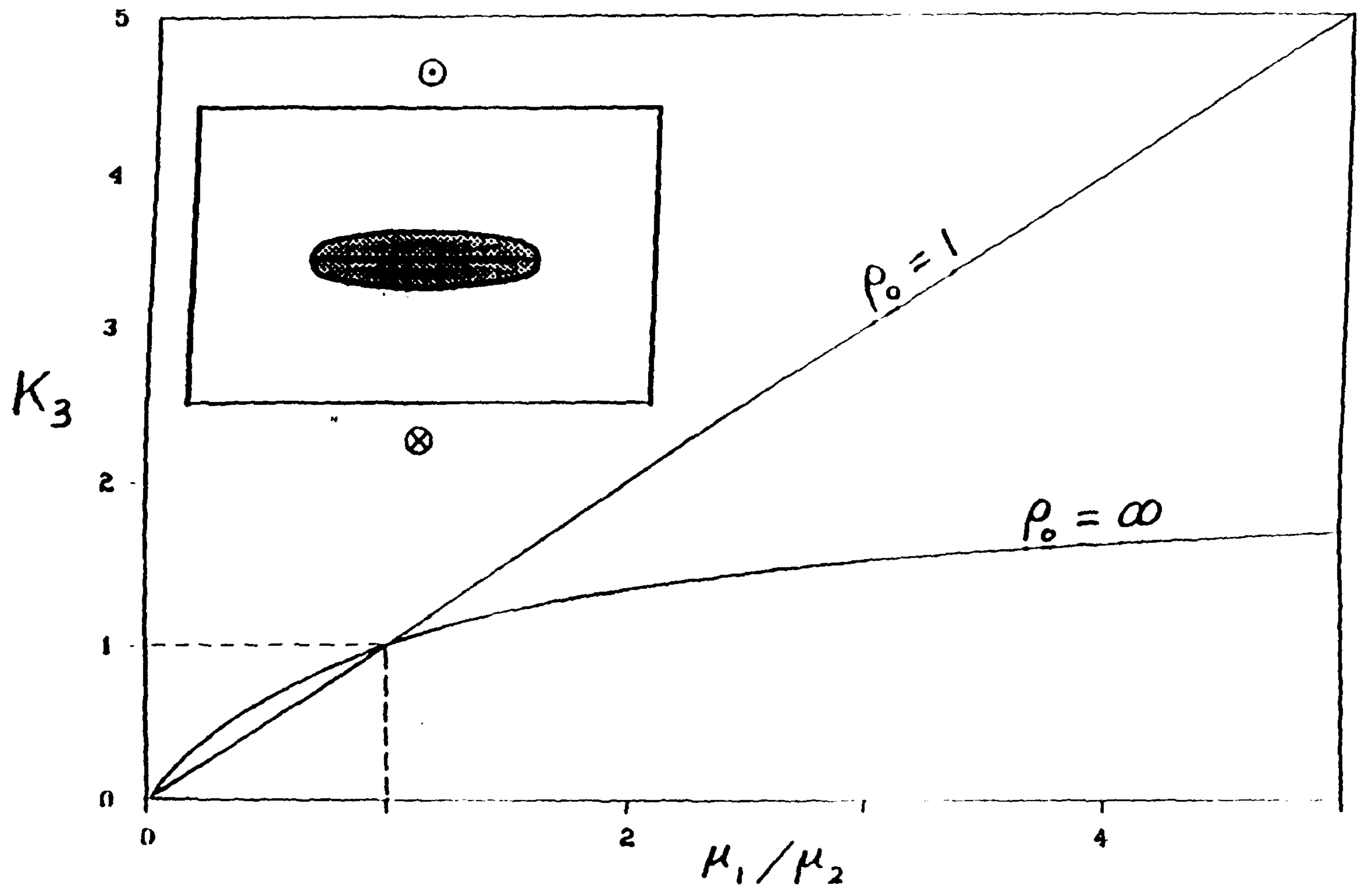


Fig. 4 2 2 K_3 as a function of μ_1/μ_2

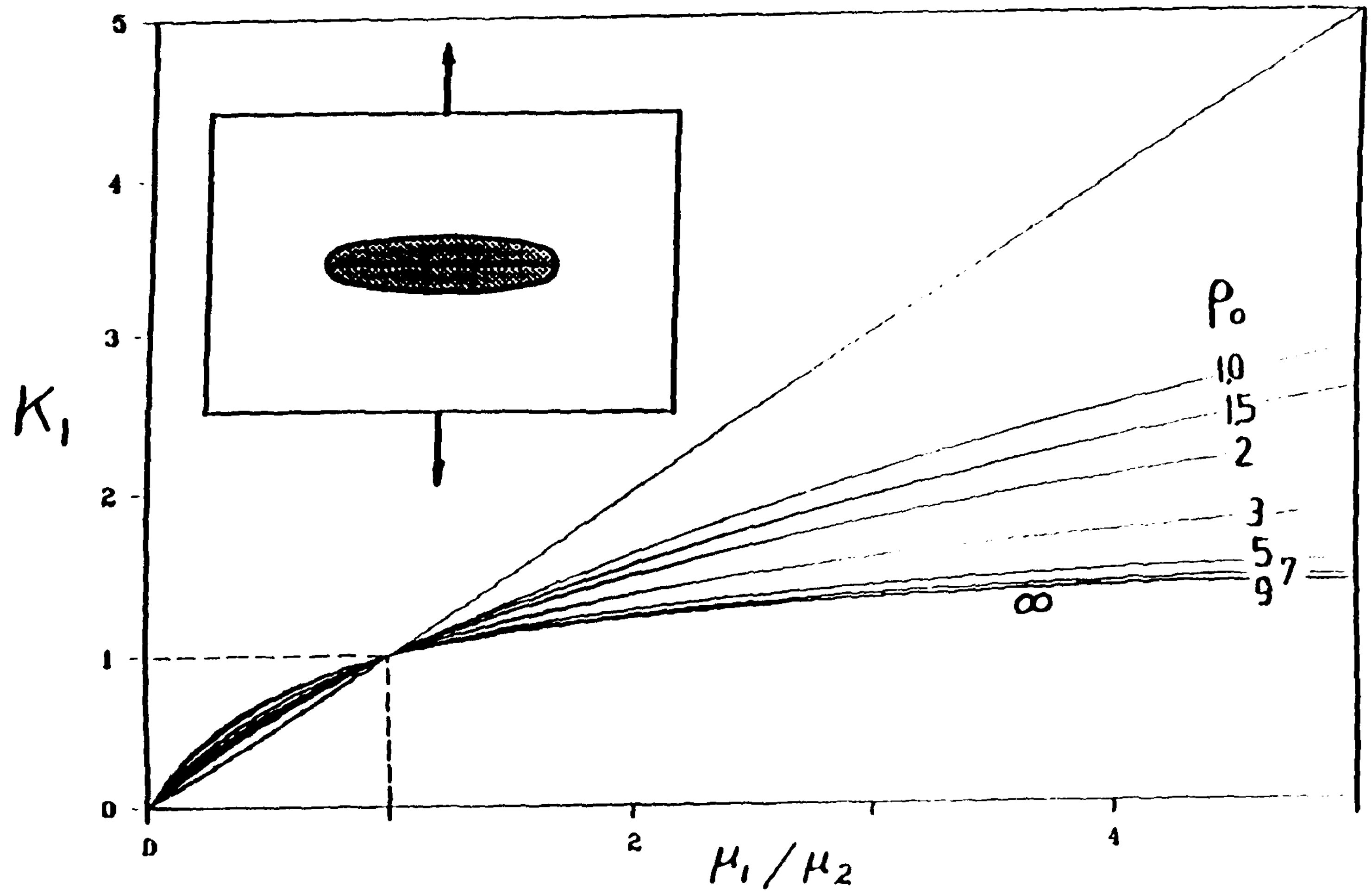


Fig. 4.2.3 K_1 as a function of μ_1/μ_2 with ρ_o as a parameter

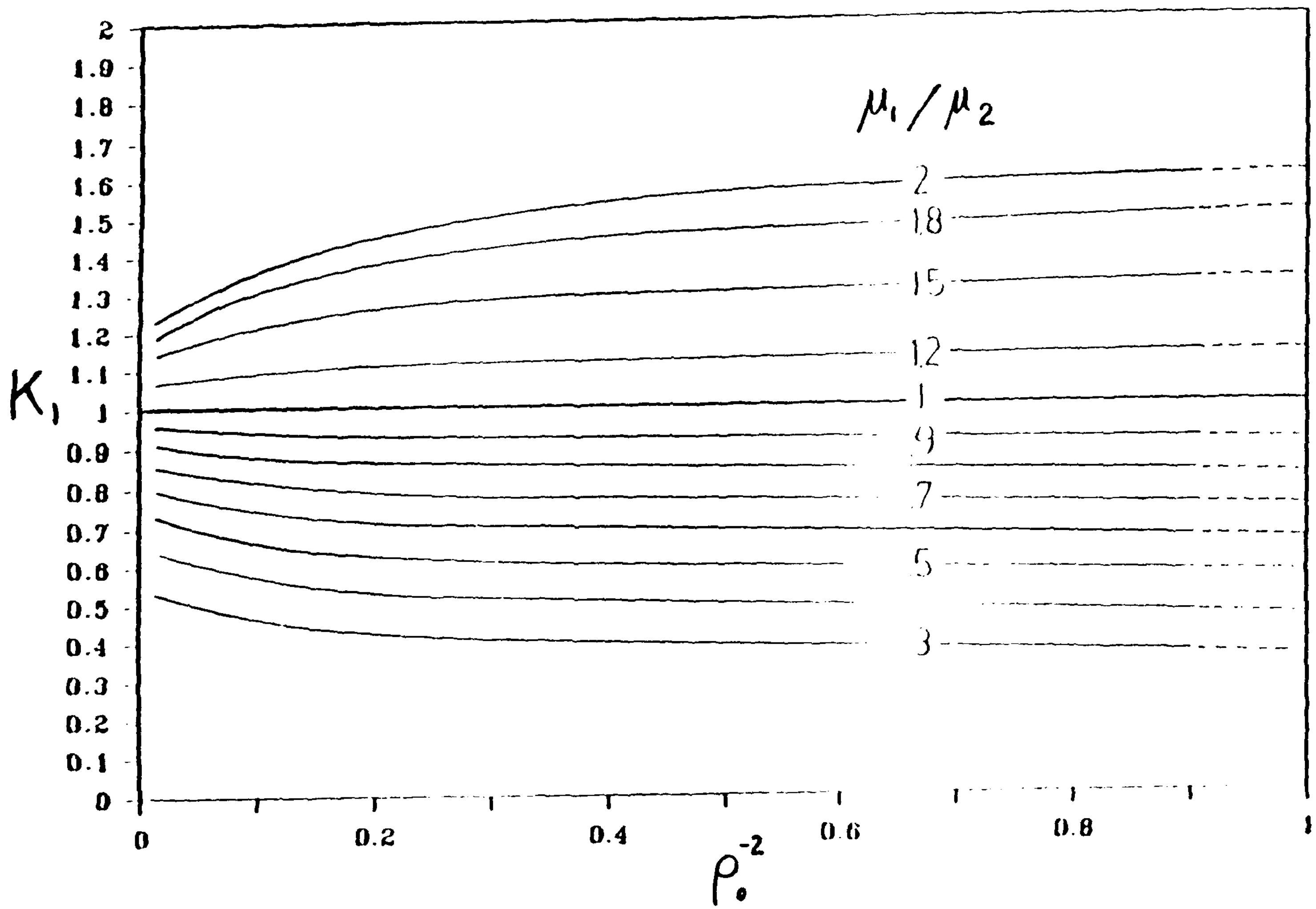


Fig. 4.2.4 K_1 as a function of ρ_o^{-2} with μ_1/μ_2 as a parameter

4.3 Asymptotic Solution of a Crack in a Vanishingly Thin Elliptic Inhomogeneity

4.3.1 Formulation

Attention is now turned to the specific situation where the parameter ρ_o used in (4.2.3) is almost equal to 1, i.e.,

$$\rho_o = 1+\epsilon\ , \quad a = \frac{1}{2}\left[\rho_o + \frac{1}{\rho_o}\right], \quad b = \frac{1}{2}\left[\rho_o - \frac{1}{\rho_o}\right], \quad a^2 - b^2 = 1\ , \tag{4.3.1}$$

where a and b are the major and minor axes of an ellipse with focal points located at ± 1. The ellipse is thin if $\epsilon << 1$. We have

$$a = 1 + \frac{1}{2}(\epsilon^2 - \epsilon^3 + \cdots)\ , \quad b = \epsilon - \frac{1}{2}(\epsilon^2 - \epsilon^3 + \cdots) \tag{4.3.2}$$

$$\delta(\epsilon) = b/a \sim O(\epsilon) \tag{4.3.3}$$

where δ is another convenient parameter. Moreover, the radius of curvature of the ellipse at $z = \pm a$ is

$$\rho_a = \frac{b^2}{a} \sim \delta^2 + 0(\epsilon^4)\ . \tag{4.3.4}$$

The ellipse is assumed to be the interface boundary, viz.,

$$C: z = z_c = x \pm i\,\frac{b}{a}(a^2 - x^2)^{1/2}\ , \qquad (x^2 < a^2) \tag{4.3.5}$$

Thus, as $\epsilon \to 0$, z_c has the expansion

$$z_c \sim x \pm i\delta(1-x^2)^{1/2} \pm i\delta^3\,\frac{1}{2}\,(1-x^2)^{-1/2} + \cdots \tag{4.3.6}$$

which is valid for $x^2 < 1$. It follows that the expansion cannot be used to satisfy the continuity conditions near $z = \pm a$. The associated expansion is termed the outer expansion

which will be presented in Section 4.3.3. The terminology has its origin in thin airfoil theory which is well–known (Van Dyke, 1975).

To remedy the shortcoming of the outer expansion, inner expansions will be constructed in Section 4.3.4. In view of the symmetry, we shall concentrate on the region near $z = a$. The scaling factor for the needed boundary–layer complex variable is dictated by (4.3.2). This complex variable is defined by (Fig. 4.3.1b)

$$\zeta = \frac{z-1}{\frac{1}{2}\,\delta^2} \tag{4.3.7}$$

The appropriate portion of C is now given by the expansion:

$$C\colon \zeta \sim \zeta_c \sim \xi \pm i2\left[(1-\xi) - \frac{\delta^2}{4}(1-\xi)^2 + \cdots\right]^{1/2} \tag{4.3.8}$$

where $\xi < 1$ and the leading term is just the parabola:

$$\zeta_c \sim \zeta_o = (1-\eta^2) + i2\eta \qquad (-\infty < \eta < +\infty)\,, \tag{4.3.9}$$

$$= \left[\cos\frac{\phi}{2}\right]^{-2} e^{i\phi} \qquad (-\pi < \phi < \pi)\,. \tag{4.3.10}$$

4.3.2 Outer Expansion

The representation (4.3.6) will be used to satisfy (2.12) and (2.13). In view of (4.3.3) and (4.3.6), we choose δ^n as the asymptotic sequence and seek the solution of a generic unknown function $F(z)$ in the form

$$F(z) \sim F(z;\delta) \sim \sum_{n=0}^{\infty} \delta^n F_n(z) \tag{4.3.11}$$

where F stands for any of the four unknown functions W_a and f_a. The value of the function $F(z_c)$ may be computed from the scheme:

$$F(z_c) \sim \sum \delta^n F_n(z_c) \sim \sum \delta^n \Big[F_n^{\pm}(x) \pm i\delta(1-x^2)^{1/2} F_n'^{\pm}(x). + \cdots\Big] \tag{4.3.12}$$

where (4.3.6) has been used for z_c.

The system of governing conditions (2.7), (2.12), (2.13) and (2.8) are now expanded in accordance with (4.3.1) and (4.3.2). The δ^0–terms are:

$$W_{10}^{\pm}(x) + W_{10}^{\mp}(x) - f_{10}^{\mp}(x) = 0 \tag{4.3.13}$$

$$W_{20}^{\pm}(x) + W_{20}^{\mp}(x) - f_{20}^{\mp}(x) = W_{10}^{\pm}(x) + W_{10}^{\mp}(x) - f_{10}^{\mp}(x) \,, \tag{4.3.14}$$

$$W_{10}^{\pm}(x) = \gamma W_{20}^{\pm}(x) + \gamma^* \Big[W_{20}^{\pm}(x) + W_{20}^{\mp}(x) - f_{20}^{\mp}(x)\Big] \,, \tag{4.3.15}$$

$$W_{20}(z) = Wz \,, \quad f_{20}(z) = fz \qquad \text{as } z \to \infty \,, \tag{4.3.16}$$

where the first three conditions apply to the interval $|x| < 1$. It follows from (4.3.13) that the left–hand side of (4.3.14) vanishes as well. It merely implies that when an interface is asymptotically near a traction–free boundary it is itself asymptotically traction–free. This is essentially the nature of the iteration associated with the outer expansion. At the same time, it is clear that the iterative mechanism can not be valid near $z = \pm a$ where the interface traction is asymptotically large in $(z-1)^{-1/2}$ as $z = a \to 1$. In any case, the solution to the above equations is

$$f_{10}(z) = \gamma f_{20}(z) = \gamma fz \,, \tag{4.3.17}$$

$$2W_{10}(z) - f_{10}(z) = \gamma\Big[2W_{20}(z) - f_{20}(z)\Big] = \gamma\sigma X(z) \,, \tag{4.3.18}$$

where

$$X(z) = (z^2-1)^{1/2}. \tag{4.3.19}$$

The δ–terms of the system of equations are then derived. They may be further simplified by the use of (4.3.17) and (4.3.18). The results are

$$W_{11}^{\pm}(x) + W_{11}^{\mp}(x) - f_{11}^{\mp}(x) = 0, \tag{4.3.20}$$

$$W_{21}^{\pm}(x) + W_{21}^{\mp}(x) - f_{21}^{\mp}(x) = (\gamma-1)\Big[(\sigma-\bar{\sigma})x \pm (f+\bar{f})i(1-x^2)^{1/2}\Big], \tag{4.3.21}$$

$$W_{11}^{\pm}(x) = \gamma W_{21}^{\pm}(x) + \gamma\gamma^*\Big[(\sigma-\bar{\sigma})x \pm (f+\bar{f})i(1-x^2)^{1/2}\Big], \tag{4.3.22}$$

$$W_{21}(z) \quad \text{and} \quad f_{21}(z) \to 0 \quad \text{as} \quad z \to \infty . \tag{4.3.23}$$

Combining the $\pm$ forms of (4.3.21), we obtain

$$f_{21}^{+}(x) - f_{21}^{-}(x) = i2(\gamma-1)(f+\bar{f})(1-x^2)^{1/2}, \tag{4.3.24}$$

$$\Big[2W_{21}^{+}(x) - f_{21}^{+}(x)\Big] + \Big[2W_{21}^{-}(x) - f_{21}^{-}(x)\Big] = 2(\sigma-\bar{\sigma})(\gamma-1)x, \tag{4.3.25}$$

which, together with (4.3.23), may be solved for f_{21} and W_{21} in terms of a Cauchy and a Hilbert problem. The solutions are:

$$f_{21}(z) = (\gamma-1)(f+\bar{f})[X(z)-z], \tag{4.3.26}$$

$$2W_{21}(z) - f_{21}(z) = -(\gamma-1)(\sigma-\bar{\sigma})[X(z)-z], \tag{4.3.27}$$

where z is a homogeneous solution of (4.3.24) and $X(z)$ a homogeneous solution of (4.3.25). They are included to meet the conditions at infinity. Substituting the above into (4.3.22) and (4.3.20), we get

$$f_{11}(z) = \gamma[(\gamma \quad -2\gamma^*)(\sigma-\bar{\sigma}) - (\gamma-1)(f+\bar{f})]z, \tag{4.3.28}$$

$$2W_{11}(z) - f_{11}(z) = \gamma[(\gamma-1+2\gamma^*)(f+\bar{f}) - (\gamma-1)(\sigma-\bar{\sigma})]X(z). \tag{4.3.29}$$

The δ^2–terms of the system of governing conditions are obtained in a similar manner. They are further simplified by the explicit lower order solutions and the results are:

$$W_{12}^{\pm}(x) + W_{12}^{\mp}(x) - f_{12}^{\mp}(x) = 0 \,, \tag{4.3.30}$$

$$\begin{aligned} W_{22}^{\pm}(x) + W_{22}^{\mp}(x) - f_{22}^{\mp}(x) &= (\gamma-1)[(2\gamma-1)f - (2\gamma+3)\bar{f}]x \\ &\mp (\gamma-1)[(2\gamma-1)f + (2\gamma-3)\bar{f}]i(1-x^2)^{1/2} - \frac{(\gamma-1)\bar{\sigma}}{\pm i(1-x^2)^{1/2}} \,, \end{aligned} \tag{4.3.31}$$

$$\begin{aligned} W_{12}^{\pm}(x) &= \gamma W_{22}^{\pm}(x) - \gamma\gamma^*\left[(f+\bar{f})x \pm (\sigma-\bar{\sigma})i(1-x^2)^{1/2}\right] \\ &+ \gamma\gamma^*\left[2(\gamma-1)(f-\bar{f})x \mp 2(\gamma-1)(f+\bar{f})i(1-x^2)^{1/2} - \frac{\bar{\sigma}}{\pm i(1-x^2)^{1/2}}\right] \,, \end{aligned} \tag{4.3.32}$$

$$W_{22}(z) \quad \text{and} \quad f_{22}(z) \to 0 \quad \text{as} \quad z \to \infty \,. \tag{4.3.33}$$

Combining the $\pm$ forms of (4.3,31) again leads to a Cauchy and Hilbert problem, and the solutions are

$$f_{22}(z) = -(\gamma-1)[(2\gamma-1)f + (2\gamma-3)\bar{f}][X(z)-z] - (\gamma-1)\bar{\sigma}X^{-1}(z) \,, \tag{4.3.34}$$

$$W_{22}(z) - f_{22}(z) = (\gamma-1)[(2\gamma+3)\bar{f} - (2\gamma-1)f][X(z)-z] + C(\gamma-1)\bar{\sigma}X^{-1}(z) \,, \tag{4.3.35}$$

where the last term of (4.3.35) is a homogeneous solution and C an arbitrary constant. The other two functions may again be determined from (4.3.32) and (4.3.30). They are

$$f_{12}(z) = 2\left\{-\gamma(\gamma-1)[3\bar{f} - (2\gamma-1)f] - \gamma\gamma^*(f+\bar{f}) + 2\gamma\gamma^*(\gamma-1)(f-\bar{f})\right\}z \,, \tag{4.3.36}$$

$$\begin{aligned} 2W_{12}(z) - f_{12}(z) &= [(C-1)(\gamma-1) - 2\gamma^*]\gamma\bar{\sigma}X^{-1}(z) \\ &+ 2\left\{\gamma(\gamma-1)[3\bar{f} - (2\gamma-1)f - \gamma\gamma^*(\sigma-\bar{\sigma})] - 2\gamma\gamma^*(\gamma-1)(f+\bar{f})\right\}X(z) \,. \end{aligned} \tag{4.3.37}$$

The procedure could be continued to include as many terms as we wish but the explicit 3–term expansion is sufficient to reveal the most important characters of the desired solution. To unveil these properties and also to facilitate matching, we need the inner expansion of the outer expansion. This is accomplished by expressing z in terms of ζ via (4.3.7) and then expanding. The results are

$$f_1 \sim \gamma f + \delta\left\{\gamma[(1-\gamma)(f+\bar{f}) - (1-\gamma-2\gamma^*)(\sigma-\bar{\sigma})] + \sum_{n=1}^{\infty} \frac{(\ )_n}{\zeta^n}\right\} + O(\delta^2)\,, \tag{4.3.38}$$

$$\begin{aligned} 2W_1 - f_1 \sim \delta\Bigg\{&\gamma\sigma\zeta^{1/2} + \gamma\bar{\sigma}[(\gamma-1)C - (\gamma-1+2\gamma^*)]\frac{1}{\zeta^{1/2}} \\ &+ \zeta^{1/2}\sum_{n=2}^{\infty}\frac{(\ )_n}{\zeta^n}\Bigg\} + O(\delta^2), \end{aligned} \tag{4.3.39}$$

$$f_2 \sim f + \delta\left\{(1-\gamma)(f+\bar{f}) + (1-\gamma)\bar{\sigma}\frac{1}{\zeta^{1/2}} + \sum_{n=2}^{\infty}\frac{(\ )_n}{\zeta^{n/2}}\right\} + O(\delta^2)\,, \tag{4.3.40}$$

$$2W_2 - f_2 \sim \delta\left\{\sigma\zeta^{1/2} + (\gamma-1)(\sigma-\bar{\sigma}) + (\gamma-1)C\bar{\sigma}\frac{1}{\zeta^{1/2}} + \sum_{n=2}^{\infty}\frac{(\ )_n}{\zeta^{n/2}}\right\} + O(\delta^2), \tag{4.3.41}$$

where the generic symbol $(\ )_n$ denotes an expression depending on the explicit forms of the outer expansion, (4.3.11). Some of them may involve arbitrary constants such as C given in (4.3.35). These constants can only be determined from matching.

The above expressions represent the properties of inner expansions, which will be derived in the following section, for $\zeta \to \infty$. It follows from (4.3.38) and (4.3.39) that

$$\begin{cases} f_1 = \text{Holomorphic function of } \zeta \\ 2W_1 - f_1 = \zeta^{1/2}[\text{Holomorphic function of } \zeta] \end{cases} \quad \text{as } \zeta \to \infty, \tag{4.3.42}$$

a property that can be directly deducted from the traction–free condition. On the other hand, (4.3.40) and (4.3.41) indicate that both f_2 and W_2 involve powers of $\zeta^{-1/2}$ as $\zeta \to \infty$. This property cannot be conceived directly from any of the governing conditions and will be needed in the series representation to be constructed in the sequel.

4.3.3 Inner Expansion

The independent variable is the boundary layer variable ζ defined by (4.3.7). Guided by (4.3.38)–(4.3.41), we shall seek the expansion of a generic unknown function $F(z)$ in the form (c.f.(4.3.11))

$$F(z) \sim F^*(\zeta,\delta) \sim \sum_{n=0}^{\infty} \delta^n F_n^*(\zeta) \tag{4.3.43}$$

where F stands for any one of the four unknowns W_a and f_a. Using $F^*(\zeta)$ to denote $F^*(\zeta,\delta)$ for simplicity, we obtain from (2.7), (2.12) and (2.13) the governing system:

$$W_1^{*+}(\xi) + W_1^{*-}(\xi) - f_1^{*-}(\xi) = 0, \qquad (\xi < 0) \tag{4.3.44}$$

$$\begin{aligned} &W_2^*(\zeta_c) + W_2^*(\bar\zeta_c) + (\zeta_c - \bar\zeta_c)\overline{W_2^{*\prime}(\zeta_c)} - f_2^*(\bar\zeta_c) \\ &\qquad = W_1^*(\zeta_c) + W_1^*(\bar\zeta_c) + (\zeta_c - \bar\zeta_c)\overline{W_1^{*\prime}(\zeta_c)} - f_1^*(\bar\zeta_c). \end{aligned} \tag{4.3.45}$$

$$\begin{aligned} W_2^*(\zeta_c) = \frac{1}{\gamma} W_1^*(\zeta_c) - \frac{\Gamma^*}{\gamma}\Big[&W_1^*(\zeta_c) + W_1^*(\bar\zeta_c) \\ &+ (\zeta_c - \bar\zeta_c)\overline{W_1^{*\prime}(\zeta_c)} - f_1^*(\bar\zeta_c)\Big], \end{aligned} \tag{4.3.46}$$

where ζ_c is defined by (4.3.8). As $\zeta \to \infty$, the functions must be matched with the inner expansions of the outer expansions (4.3.38)–(4.3.41).

The δ^n–order conditions may be derived by using the following substitution:

$$F^*(\zeta_c) \sim \sum \delta^n F_n^*(\zeta_c) \sim \sum \delta^n \left[F_n^*(\delta_o) \mp \delta^2 i \, \frac{1}{4}\,(1-\zeta)^{3/2} F_n^{*\prime}(\zeta_o) + \cdots\right] \qquad (4.3.47)$$

where ζ_o is defined by (4.3.9). For the δ^o–terms, the choice

$$W_{10}^*(\zeta) = \frac{1}{2}\, f_{10}^*(\zeta) = \gamma W_{20}^*(\zeta) = \frac{\gamma}{2}\, f_{20}^*(\zeta) = \frac{\gamma}{2}\, f \qquad (4.3.48)$$

satisfies all the conditions and yields no useful information. The δ–order system governs the desired asymptotic limit $(\delta \to 0)$ and the subsequent discussions will be concentrated on the solution of the system.

We begin by writing

$$f_{11}^*(\zeta) = \gamma\left[(1-\gamma)(f+\bar{f}) - (1-\gamma-2\gamma^*)(\sigma-\bar{\sigma})\right] + \gamma\sigma h_1(\zeta)\,, \qquad (4.3.49)$$

$$2W_{11}^*(\zeta) - f_{11}^*(\zeta) = \gamma\sigma\zeta^{1/2}\left[1 + H_1(\zeta)\right], \qquad (4.3.50)$$

$$f_{21}^*(\zeta) = (1-\gamma)(f+\bar{f}) + \sigma h_2(\zeta)\,, \qquad (4.3.51)$$

$$W_{21}^*(\zeta) = \frac{1}{2}\,(1-\gamma)\left[(f+\bar{f}) - (\sigma-\bar{\sigma})\right] + \frac{\sigma}{2}\,\zeta^{1/2} + \sigma H_2(\zeta)\,. \qquad (4.3.52)$$

Comparing the above with (4.3.38)–(4.3.41), we conclude that the explicitly listed terms are matched and

$$h_a(\zeta)\,, \quad H_a(\zeta) = 0 \quad \text{as} \quad \zeta \to \infty\,. \qquad (4.3.53)$$

Since the geometry is symmetric and the loading symmetry is maintained by the factor $\mathcal{J}$ defined by (2.10), we conclude from the above that the functions h_a and H_a, signified by a generic symbol F, must satisfy the condition

$$\overline{F(\zeta)} = F(\bar{\zeta}) \quad \text{for } h_a \text{ and } H_a . \tag{4.3.54}$$

Finally, (4.3.44) is identically satisfied if h_1 and H_1 are holomorphic in D_1 in the ζ-plane. We must now determine h_a and H_a to meet the two remaining conditions (4.3.45) and (4.3.46).

Before proceeding, we establish a few identities associated with the function ζ_o given by (4.3.9). Since

$$\zeta_o^{1/2} = 1+i\eta , \tag{4.3.55}$$

we may express $\bar{\zeta}_o^{1/2}$ in terms of $\zeta_o^{1/2}$, i.e.,

$$\bar{\zeta}_o^{1/2} = \Psi(\zeta_o) = 2-\zeta_o^{1/2} \tag{4.3.56}$$

where $\Psi(\zeta)$ is holomorphic in the $\zeta(=\zeta_1+i\zeta_2)$-plane with a cut along the ζ_1 axis, the crack boundary. We have

$$\Psi\left[\Psi^2(\zeta)\right] = \zeta^{1/2} , \tag{4.3.57}$$

$$\frac{1}{\Psi(\zeta)} = -\frac{4}{\zeta-4} \quad \text{as } \zeta \to 4 , \tag{4.3.58}$$

$$\frac{1}{4}\frac{\zeta_o - \bar{\zeta}_o}{\bar{\zeta}_o^{1/2}} = \frac{1}{\Psi(\zeta_o)} - 1 . \tag{4.3.59}$$

Substituting (4.3.49), (4.3.50) and (4.3.52) into (4.3.46) and using (4.3.54), (4.3.56) and (4.3.59), we find that the condition is satisfied if

$$H_2(\zeta) = H_2^*\{H_1,h_1\} = \frac{1}{2}\left[\zeta^{1/2}H_1(\zeta) + h_1(\zeta)\right] - \gamma^*\left\{\frac{1}{2}\left[h_1(\zeta) - h_1\left[\Psi^2(\zeta)\right]\right.\right.$$

$$\left. + \zeta^{1/2}H_1(\zeta) + \Psi(\zeta)H_1\left[\Psi^2(\zeta)\right]\right] + \sigma^*\left[\zeta^{1/2}-1\right]\left[\frac{H_1(\Psi^2(\zeta))}{\Psi(\zeta)}\right.$$

$$\left.\left. + 2\Psi(\zeta)H_1'\left[\Psi^2(\zeta)\right] + 2h_1'\left[\Psi^2(\zeta)\right]\right] + \frac{\sigma^*}{\Psi(\zeta)}\right\}, \tag{4.3.60}$$

where

$$\sigma^* = \frac{\bar{\sigma}}{\sigma} = \begin{cases} +1 & \text{if} \quad \sigma = \sigma_{22} \\ -1 & \phantom{\text{if}} \quad \sigma = i\sigma_{12} \end{cases}. \tag{4.3.61}$$

Substituting the above into (4.3.45) and using (4.3.56) and (4.3.57), we find that the condition is satisfied if

$$h_2(\zeta) = h_2^*\{H_1,h_1\} = H_2(\zeta) = 4\sigma^*\left[\zeta^{1/2}-1\right]H_2'(\zeta) + \frac{1}{2}(1-\gamma-\gamma^*)\left[\frac{2\sigma^*}{\zeta^{1/2}}\right.$$

$$\left. + \Psi(\zeta)H_1\left[\Psi^2(\zeta)\right] + h_1\left[\Psi^2(\zeta)\right]\right] - (\gamma+\gamma^*)\left\{\frac{1}{2}\left[\zeta^{1/2}H_1(\zeta) - h_1(\zeta)\right]\right.$$

$$\left. + \sigma^*\left[1-\zeta^{1/2}\right]\left[\frac{H_1(\zeta)}{\zeta^{1/2}} + 2\zeta^{1/2}H_1'(\zeta) + 2h_1'(\zeta)\right]\right\}. \tag{4.3.62}$$

The above two expressions $H_2^*\{H_1,h_1\}$ and $h_2^*\{H_1,h_1\}$ may be taken as two calculating machines which convert the input h_1 and H_1 into the products h_2 and H_2 in such a way that the continuity conditions are identically satisfied. For a pair of chosen H_1 and h_1 satisfying (4.3.53), H_2 and h_2 calculated from (4.3.61) and (4.3.62) also satisfy (4.3.53). The only condition that is left to be enforced is that h_a and H_a must be holomorphic in D_a in the ζ-plane (c.f. Fig. 4.3.1b).

Let us begin by setting $H_1 = h_1 = 0$. Then,

$$H_2^*\{0,0\} = -\frac{\gamma^*\sigma^*}{\Psi(\zeta)} \tag{4.3.63}$$

$$h_2^*\{0,0\} = \frac{(1-\gamma-\gamma^*)\sigma^*}{\zeta^{1/2}} - \frac{\gamma^*\sigma^*}{\Psi(\zeta)} + \frac{2\gamma^*(\zeta^{1/2}-1)}{\zeta^{1/2}\left[2-\zeta^{1/2}\right]^2} \tag{4.3.64}$$

which have poles at $\zeta = 4$, i.e.,

$$H_2^*\{0,0\} = P_{20}^*(\zeta) = \frac{4\gamma^*\sigma^*}{\zeta-4} \qquad (\zeta \to 4) \tag{4.3.65}$$

$$h_2^*\{0,0\} = p_{20}^*(\zeta) = \frac{4\gamma^*\sigma^*}{\zeta-4} + \frac{16\gamma^*}{(\zeta-4)^2} \qquad (\zeta \to 4) \tag{4.3.66}$$

Thus, $H_1 = h_1 = 0$ cannot be the correct choice. To obtain the complete solution, we write

$$\begin{aligned} H_1(\zeta) &= R_{10}(\zeta) \\ h_1(\zeta) &= r_{10}(\zeta) \\ H_2(\zeta) &= H_2^*\{0,0\} + \left[R_{20}(\zeta) - P_{20}^*(\zeta)\right] \\ h_2(\zeta) &= h_2^*\{0,0\} + \left[r_{20}(\zeta) - p_{20}^*(\zeta)\right] \end{aligned} \tag{4.3.67}$$

where $R_{\alpha o}$ and $r_{\alpha o}$ are holomorphic in D_α. The following series are assumed

$$\begin{aligned} r_{10}(\zeta) &= \sum_{n=1}^{\infty} \frac{a_n}{(\zeta-4)^n}, \qquad R_{10} = \sum_{n=1}^{\infty} \frac{b_n}{(\zeta-4)^n}, \\ r_{20}(\zeta) &= \sum_{n=1}^{\infty} \frac{A_n}{\zeta^{n/2}}, \qquad R_{20} = \sum_{n=1}^{\infty} \frac{B_n}{\zeta^{n/2}}, \end{aligned} \tag{4.3.68}$$

which conform with the requirement at $\zeta = \infty$ characterized by (4.3.38)–(4.3.41). The constants a_n, b_n, A_n and B_n must be chosen in such a way that the substitutions (c.f. (4.3.49)–(4.3.52))

$$
\begin{aligned}
&f_1^*:\ \gamma\sigma r_{10}(\zeta)\\
&2W_1^* - f_1^*:\ \gamma\sigma\zeta^{1/2}R_{10}(\zeta)\\
&f_2^*:\ \sigma\left[r_{20}(\zeta) - p_{20}^*(\zeta)\right]\\
&W_2^*:\ \sigma\left[R_{20}(\zeta) - P_{20}^*(\zeta)\right]
\end{aligned}
\tag{4.3.69}
$$

satisfy the two continuity conditions (4.3.45) and (4.3.46). Since all terms are bounded, the standard Fast Fourier Transform algorithm may be applied. This provides a numerical scheme for the direct computation of the needed asymptotic solution as opposed to the numerically extrapolated result of section 4.2 .

Let K_I and K_{II} be the SIF's. They may be normalized by the factors $\sigma_{22}\sqrt{\pi}$ and $\sigma_{12}\sqrt{\pi}$, i.e.,

$$
K_1 = K_I/\sigma_{22}\sqrt{\pi}\,, \qquad K_2 = K_{II}/\sigma_{12}\sqrt{\pi} \tag{4.3.70}
$$

where K_1 and K_2 depend on the small parameter δ. As $\delta \to 0$, they tend to finite limits which may be obtained from (4.3.50). The results are

$$
\begin{aligned}
K_1 &= \gamma\left[1 + H_1(0)\right] \qquad && \left\{\sigma^* = \frac{\bar{\sigma}}{\sigma} = +1\right\},\\
K_2 &= \gamma\left[1 + H_1(0)\right] \qquad && \left\{\sigma^* = \frac{\bar{\sigma}}{\sigma} = -1\right\},
\end{aligned}
\tag{4.3.71}
$$

where

$$
H_1(0) = R_{10}(0) = \sum_{n=1}^{\infty} \frac{b_n}{(\zeta-4)^n}\,. \tag{4.3.72}
$$

The above analysis establishes the basis for the general solution which will be discussed in section 4.4.

4.3.4 An Approximate Analytic Solution

It appears that an approximate but explicit solution may be constructed for the special confocal geometry. This is accomplished in the following manner. Another way of removing the poles defined by (4.3.65) and (4.3.66) is to define H_1 and h_1 as follows

$$H_1(\zeta) = P_{11}^*(\zeta) = P_1(\zeta) = \frac{P_{11}}{\zeta-4} + \frac{P_{12}}{(\zeta-4)^2} \quad , \tag{4.3.73}$$

$$h_1(\zeta) = p_{11}^*(\zeta) = p_1(\zeta) = \frac{p_{11}}{\zeta-4} + \frac{p_{12}}{(\zeta-4)^2} \quad , \tag{4.3.74}$$

where P_{11}, P_{12}, p_{11} and p_{12} are constants and both expressions are holomorphic in D_1 in the ζ-plane.

Substituting the above into (4.3.60) and (4.3.62) and requiring that the calculated H_2 and h_2 are free of poles at $\zeta = 4$, we obtain

$$2P_1(\zeta) + p_1(\zeta) + \frac{8\gamma^*\sigma^*[1+H_1(0)]}{1-\gamma^*}\frac{1}{\zeta-4} = 0 \, , \tag{4.3.75}$$

$$(2-\sigma^*)P_1(\zeta) - p_1(\zeta) - 4\Big[2P_1'(\zeta) + p_1'(\zeta)\Big] = 0 \, , \tag{4.3.76}$$

and hence

$$P_1(\zeta) = \frac{\Gamma}{4-\sigma^*}\left[-\frac{1}{\zeta-4} + \frac{4}{(\zeta-4)^2}\right] \tag{4.3.77}$$

$$p_1(\zeta) = \frac{\Gamma}{4-\sigma^*}\left[-\frac{\sigma^*-2}{\zeta-4} + \frac{8}{(\zeta-4)^2}\right] \tag{4.3.78}$$

where

$$\Gamma = \frac{8\gamma^*\sigma^*}{1-\gamma^*}\Big[1 + H_1(0)\Big] \, . \tag{4.3.79}$$

The newly generated H_2 and h_2 are now free of poles at $\zeta = 4$, but new poles are generated at $\zeta = 16$. This is so because $H_1(\Psi^2(\zeta))$ now include powers of

$$\frac{1}{\Psi^2(\zeta)-4} = -\frac{2}{\zeta-16} \qquad \text{as} \quad \zeta \to 16 . \tag{4.3.80}$$

Removing the poles at $\zeta = 16$ will lead to new poles at $\zeta = 36$.

The process may be continued for a chosen number of times so that

$$H_1 = P^*_{1N}(\zeta) = \sum_{n=1}^{N} P_n(\zeta) , \tag{4.3.81}$$

$$h_1 = p^*_{1N}(\zeta) = \sum_{n=1}^{N} p_n(\zeta) , \tag{4.3.82}$$

where

$$P_n(\zeta) = \sum_{m=1}^{2n} \frac{P_{nm}}{\left[\zeta-(2n)^2\right]^m} , \tag{4.3.83}$$

$$p_n(\zeta) = \sum_{m=1}^{2n} \frac{p_{nm}}{\left[\zeta-(2n)^2\right]^m} , \tag{4.3.84}$$

which may be expressed in terms of $P_{n-1}(\zeta)$ and $p_{n-1}(\zeta)$ via recurrence formulas. The derivation is straightforward but lengthy and is not included in the thesis.

The functions generated by substituting (4.3.81) and (4.3.82) into (4.3.60) and (4.3.62) have poles at $\zeta = [2(N+1)]^2$, i.e.,

$$\left.\begin{aligned} H^*_2\left\{P^*_{1N}(\zeta), p^*_{1N}(\zeta)\right\} &= P^*_{2N}(\zeta) \\ & \qquad \text{as } \zeta \to [2(N+1)]^2 \\ h^*_2\left\{P^*_{1N}(\zeta), p^*_{1N}(\zeta)\right\} &= p^*_{2N}(\zeta) \end{aligned}\right\} \tag{4.3.85}$$

where P^*_{2N} and p^*_{2N} have forms similar to those given by (4.3.81) and (4.3.82). Thus, (4.3.81) and (4.3.82), together with the generated H_2 and h_2, cannot be the correct solution. The complete solution must be of the form

$$
\begin{aligned}
H_1(\zeta) &= P^*_{1N}(\zeta) + R_{1N}(\zeta)\,, \\
h_1(\zeta) &= p^*_{1N}(\zeta) + r_{1N}(\zeta)\,, \\
H_2(\zeta) &= H^*_2\left\{P^*_{1N}, p^*_{1N}\right\} + \left[R_{2N}(\zeta) - P^*_{2N}(\zeta)\right], \\
h_2(\zeta) &= h^*_2\left\{P^*_{1N}, p^*_{1N}\right\} + \left[r_{2N}(\zeta) - p^*_{2N}(\zeta)\right],
\end{aligned}
\tag{4.3.86}
$$

where $R_{\alpha N}$ and $r_{\alpha N}$ play the roles of $R_{\alpha 0}$ and $r_{\alpha 0}$ in (4.3.67) and must be determined in a similar manner. In fact, the series (4.3.68) may be taken as the solution for the new unknowns. The input to this problem is provided by P^*_{2N} and p^*_{2N} which have poles of order $2N$ at $\zeta = [2(N+1)]^2$. Noting that the nearest interface point is at $\zeta = 1$, we conclude that the contributions of $R_{\alpha N}$ and $r_{\alpha N}$ are perhaps small for large N. This, however, is not proven in the thesis.[1] Taking the statement for granted, we have

$$
\begin{aligned}
&H_1(\zeta) \approx P^*_{1N}(\zeta)\,, \qquad h_1(\zeta) \approx p^*_{1N}(\zeta)\,, \\
&H_2(\zeta) \approx H^*_2\left\{P^*_{1N}, P^*_{1N}\right\} - P^*_{2N}(\zeta)\,, \\
&h_2(\zeta) \approx h^*_2\left\{P^*_{1N}, p^*_{1N}\right\} - p^*_{2N}(\zeta)\,.
\end{aligned}
\tag{4.3.87}
$$

For a two–term approximation, P_1 and p_1 are given by (4.3.77) and (4.3.78). A lengthy computation yields the following expressions for P_2 and p_2

$$
\begin{aligned}
P_2(\zeta) = \frac{2\Gamma}{(4-\sigma^*)(16-3\sigma^*)} \Bigg\{ &\left[\frac{2(1-\gamma-\gamma^*)\sigma^*}{\gamma+\gamma^*} + \frac{4(2+\sigma^*)\gamma^*}{1\cdot\gamma^*}\right]\frac{1}{(\zeta-16)} \\
&- \left[\frac{64(1-\gamma-\gamma^*)}{\gamma+\gamma^*} + \frac{48\gamma^*(2+3\sigma^*)}{1-\gamma^*}\right]\frac{1}{(\zeta-16)^2} + \frac{384\gamma^*(3+11\sigma^*)}{1-\gamma^*}\frac{1}{(\zeta-16)^3} \\
&- \frac{(3\cdot 16)^3\gamma^*}{1-\gamma^*}\frac{1}{(\zeta-16)^4}\Bigg\}
\end{aligned}
\tag{4.3.88}
$$

[1]The recurrence formulas associated with (4.3.83) and (4.3.84), together with the explicit forms of (4.3.85), are needed for such a proof.

$$p_2(\zeta) = -\frac{3\sigma^*}{4} P_2(\zeta) + \frac{\Gamma}{2(4-\sigma^*)} \left\{ \left[\frac{2(1-\gamma-\gamma^*)\sigma^*}{\gamma+\gamma^*} + \frac{4(2+\sigma^*)\gamma^*}{1-\gamma^*}\right] \frac{1}{(\zeta-16)} \right.$$

$$+ \left[\frac{64(1-\gamma-\gamma^*)}{\gamma+\gamma^*} + \frac{48\gamma^*\sigma^*}{1-\gamma^*}\right] \frac{1}{(\zeta-16)^2} + \frac{384\gamma^*(11\sigma^*-3)}{1-\gamma^*} \frac{1}{(\zeta-16)^3}$$

$$\left. - \frac{(3\cdot 16)^3\gamma^*}{1-\gamma^*} \frac{1}{(\zeta-16)^4} \right\}, \tag{4.3.89}$$

where Γ is defined by (4.3.79). It follows from the approximation

$$H_1(0) \approx P_1(0) + P_2(0) \tag{4.3.90}$$

that

$$\frac{H_1(0)}{1+H_1(0)} \approx \frac{\gamma^*\sigma^*}{(1-\gamma^*)(4-\sigma^*)} \left\{ 4 - \frac{1}{16-3\sigma^*} \left[\frac{2(2+\sigma^*)(1-\gamma-\gamma^*)}{\gamma+\gamma^*} \right.\right.$$

$$\left.\left. + \frac{(91+59\sigma^*)\gamma^*}{2(1-\gamma^*)}\right]\right\}. \tag{4.3.91}$$

The approximate expressions for the asymptotic values of K_1 and K_2 (4.3.71) are

$$K_1 \simeq \frac{3\gamma(1-\gamma^*)}{3-7\gamma^* + \frac{\gamma^*}{13}\left[\frac{6(1-\gamma-\gamma^*)}{\gamma+\gamma^*} + \frac{75\gamma^*}{1-\gamma^*}\right]}, \tag{4.3.92}$$

$$K_2 \simeq \frac{5\gamma(1-\gamma^*)}{5-\gamma^* - \frac{\gamma^*}{19}\left[\frac{2(1-\gamma-\gamma^*)}{\gamma+\gamma^*} + \frac{16\gamma^*}{1-\gamma^*}\right]}. \tag{4.3.93}$$

4.3.5 Rsults And Discussion

We shall restrict our attention to the range $0 < \mu_1/\mu_2 < 1$ to reflect our interest in the physical situation of crack damage interaction. The confocal nature of the problem allows a series solution in a transformed plane. Such a series solution was obtained in section 4.2. The convergence of the needed matrix inversion becomes extremely slow for $\epsilon < 0.05$. Nevertheless, the $\epsilon \to 0$ limit was accurately extrapolated from the extensive numerical results. This limit is reproduced for K_1 for the case $\nu_1 = \nu_2 = 0.2$ in Fig.4.3.2 where the numerical results generated by (4.3.68) is also shown to be in agreement with the series solution. To substantiate the accuracy of our two-term approximate solution, (4.3.92) and (4.3.93), K_1 is also plotted in Fig.4.3.2. The agreement is almost perfect.

Our formulation indicates that the solution to the problem depends explicitly on the two composite parameters γ and γ^* defined by (2.14). It is therefore desirable to present the SIF's in terms of them. The choice of the two composite parameters in a typical two-phase problem is not unique (Dundurs, 1969). The parameters γ and γ^* appear naturally in our problem. Moreover, exact solution may be obtained for the case $\gamma^* = 0$, (4.2.47). Still, the range of the two parameters may be easily determined from Dundurs' discussion. In terms of γ and γ^*, Dundurs' α and β parameters are

$$\alpha = \frac{1-\gamma}{1+\gamma}, \qquad \beta = \frac{1-\gamma}{1+\gamma} - \frac{2\gamma^*}{1+\gamma}, \tag{4.3.94}$$

which may be inverted to yield

$$\gamma = \frac{1-\alpha}{1+\alpha}, \qquad \gamma^* = \frac{\alpha-\beta}{1+\alpha}. \tag{4.3.95}$$

Dundurs' $\alpha-\beta$ plane, together with the admissible ranges of α and β, are reproduced in Fig. 4.3.3. Also plotted in the figure are the constant γ and γ^* lines. Thus the ranges of γ and γ^* may be straightforwardly calculated, i.e.,

$$\frac{1}{2}(1-4\gamma^*) < \gamma < 2(1-2\gamma^*) \qquad \text{for } 0 \leq \gamma^* \leq \frac{1}{4}$$
$$0 < \gamma < 2(1-2\gamma^*) \qquad \text{for } \frac{1}{4} \leq \gamma^* \leq \frac{1}{2} \tag{4.3.96}$$

which covers the range of interest set by $0 \leq \mu_1/\mu_2 \leq 1$. Results pertaining to (4.3.92) and (4.3.93) are plotted in Fig.4.3.4 as functions of γ with γ^* as a parameter. It is felt that results for all inhomogeneity problems should be presented in terms of γ and γ^* in the fashion characterized by Fig.4.3.4.

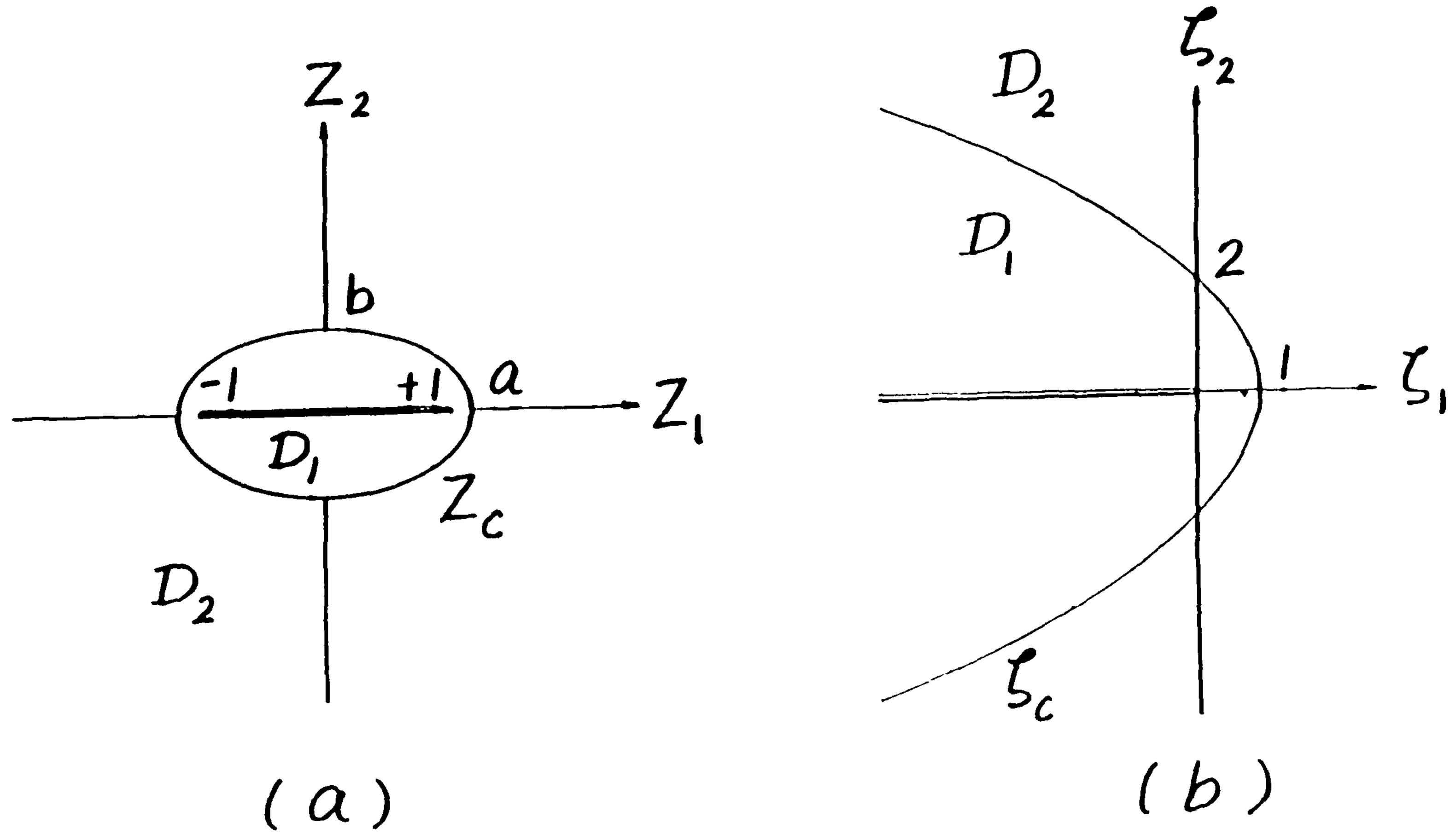


Fig. 4.3.1 A crack in a vanishingly thin elliptic inhomogeneity
a) Configuration in the physical z-plane $b/a \to 0$
b) Parabolic nose in the boundary layer ζ-plane

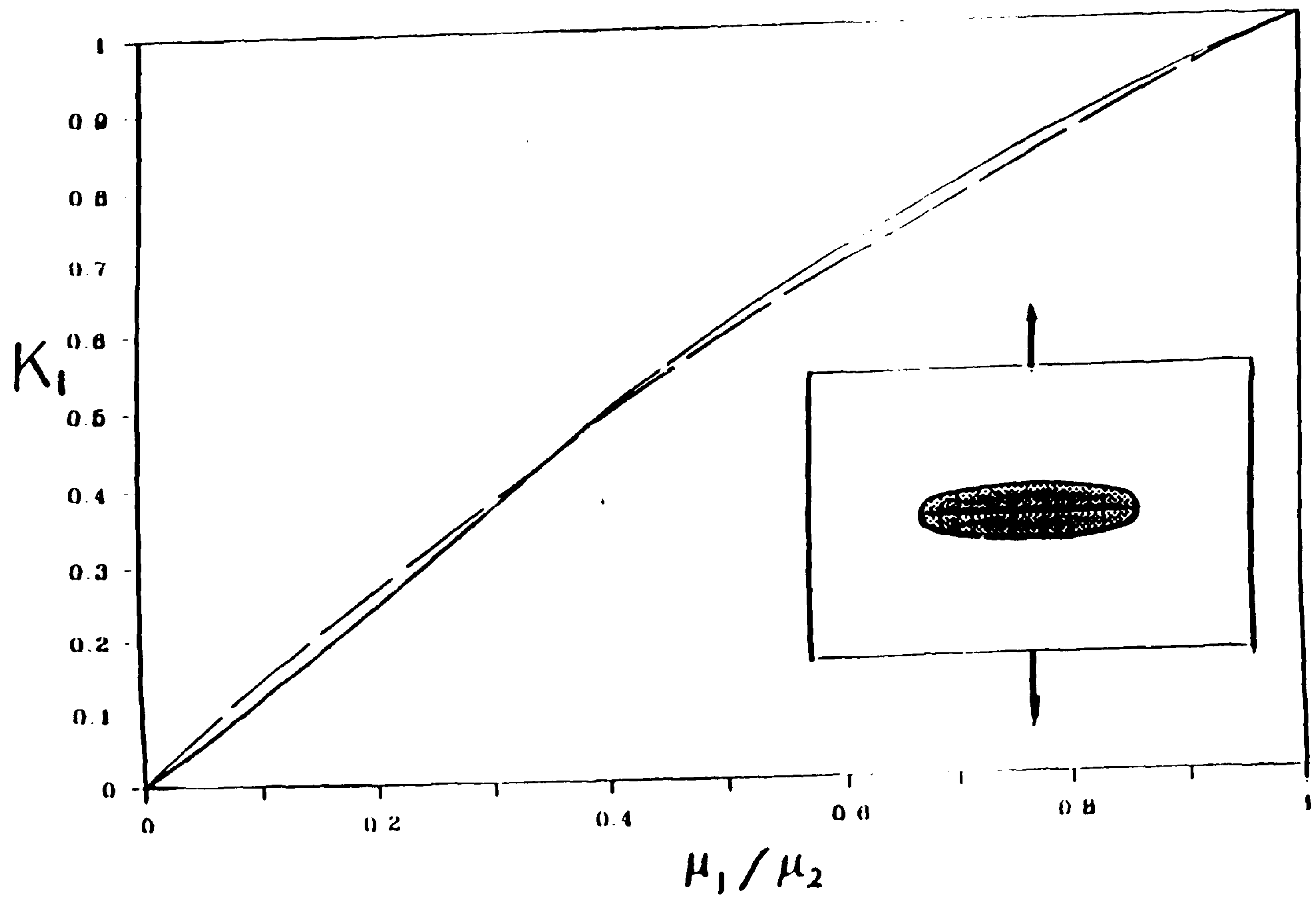


Fig. 4.3.2 K_1 as a function of μ_1/μ_2 for $b/a \to 0$ and $\nu_1 = \nu_2 = 0.2$ (plane strain)

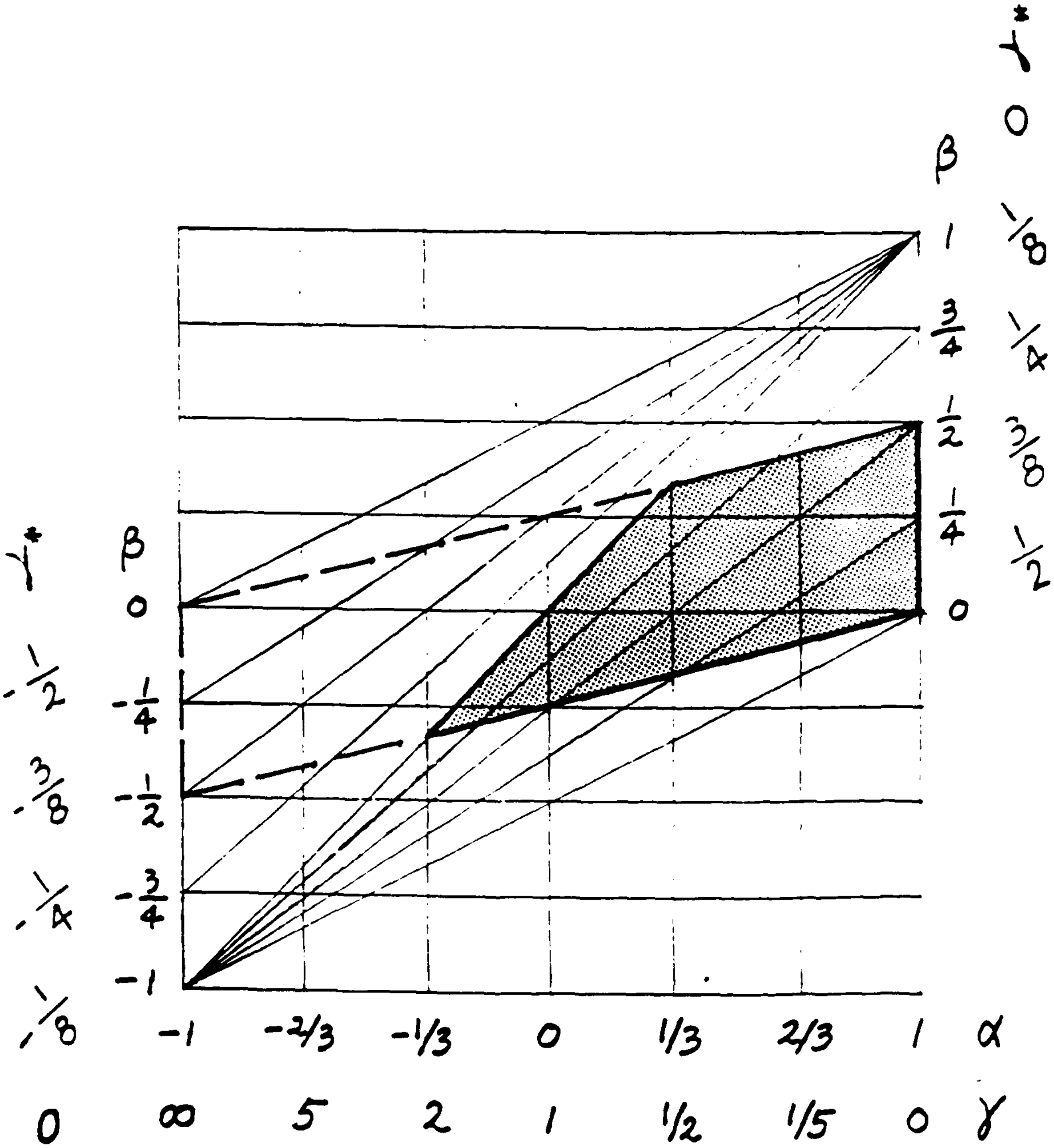


Fig. 4.3.3 Admissible region in terms of composite elastic constants (α,β) and (γ,γ^*)

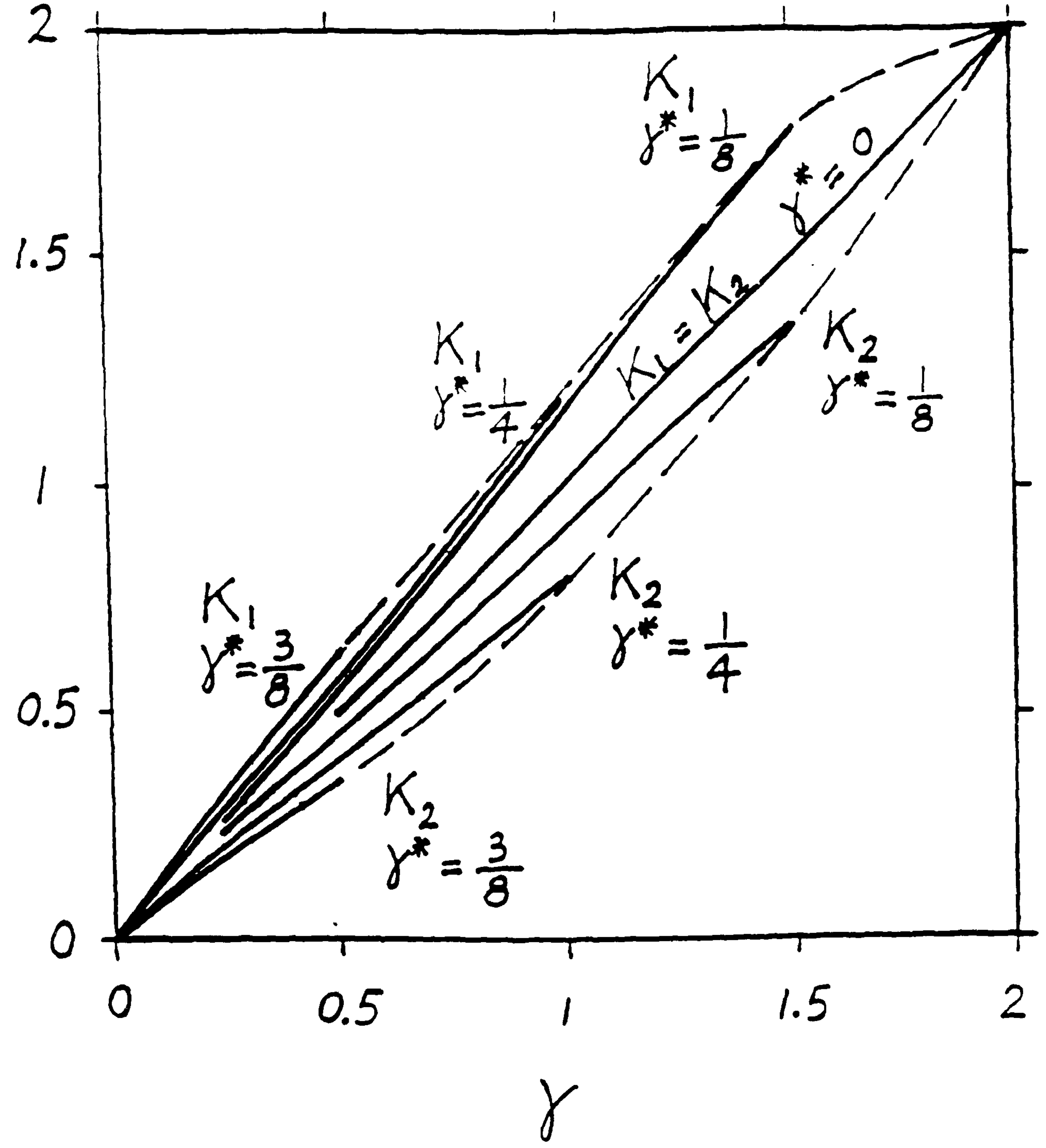


Fig. 4.3.4 K_1 and K_2 for all admissible combinations of γ and γ^*

4.4 A Crack in a Thin Inhomogeneity of Arbitrary Shape

4.4.1. Introduction

Equiped with the benchmark results of section 4.2 and the analytic information revealed by the asymptotic analysis of section 4.3, we are now ready to construct the asymptotic solution for problem II where the inhomogeneity is thin but is otherwise arbitrary. The geometric configuration is examplified by the right–hand side of (2.16). It is noted that the nose dimension implied by the second expression of (2.16) is of the order of the thickness , ϵ , of the inhomogeneity. This choice is made to reflect the physically observed damage–zone shape reported by Chudnovsky, Moet and Botsis (1987). Mathematically, though, the situation is in contrast to that of a confocal geometry where the nose dimension is of the order of ϵ^2, (4.3.2) and (4.3.7). In this connection and also in terms of mathematical considerations, the confocal situation is even harder to solve than the general case, as the former requires the satisfying of boundary condition on a parabola in the boundary layer variable.

Our final product is an asymptotically accurate series solution which is valid in the neighbourd of a crack tip for both the inhomogeneity and the medium. As a result, the solution may be manupilated to yield any and all desired physical quantities of physical importance. For example, the various generalized Eshelby forces along the interface boundary may be straightforwardly computed. These quantities will be discussed in Chapter V.

4.4.2 Formulation

We now turn to the situation depicted in Fig 1.2. Geometrically speaking, it is the elasticity counterpart of the thin airfoil problem (Von Dyke, 1975). The problem is of course much more involved as it involves four complex functions. The geometric description is given by (2.16) where, in general, $y^{\pm}(x) = y^{\pm}(x,\epsilon)$ may be assumed to have a given asymptotic expansion. We are mainly concerned with the first order solution and such a generalization is not pursued in our analysis. The mathematical problem consists of (2.7), (2.12), (2.13) and (2.8), and is recapitulated as follows:

$$W_1^{\pm}(x,\epsilon) + W_1^{\mp}(x,\epsilon) - f_1^{\mp}(x,\epsilon) = 0 \quad \text{for } |x| < 1, \tag{4.4.1}$$

$$W_2(z_c;\epsilon) + W_2(\bar{z}_c;\epsilon) + (z_c-\bar{z}_c)\overline{W_2'(z_c;\epsilon)} - f_2(\bar{z}_c;\epsilon)$$
$$= W_1(z_c;\epsilon) + W_1(\bar{z}_c;\epsilon) + (z_c-\bar{z}_c)\overline{W_1'(z_c;\epsilon)} - f_1(\bar{z}_c;\epsilon), \tag{4.4.2}$$

$$W_1(z_c;\epsilon) = \gamma W_2(z_c;\epsilon)$$
$$+ \gamma^*\left[W_2(z_c;\epsilon) + W_2(\bar{z}_c;\epsilon) + (z_c-\bar{z}_c)\overline{W_2'(z_c;\epsilon)} - f_2(\bar{z}_c;\epsilon) \right], \tag{4.4.3}$$

$$W_2(z;\epsilon) = Wz, \quad , f_2(z;\epsilon) = fz \qquad \text{as } z \to \infty. \tag{4.4.4}$$

4.4.3 Outer Expansion.

We seek the expansion of a generic unknown function $F(z)$ in the form.

$$F(z;\epsilon) \sim F_o(z) + \delta_1(\epsilon)F_1(z) + \delta_2(\epsilon)F_2(z) + \ldots.. \tag{4.4.5}$$

where $\delta_n(\epsilon)$ is an asymptotic sequence that may depend on $z_c(x;\epsilon)$. The value of the

function $F(z_c;\epsilon)$ may be computed from the scheme

$$F(z_c;\epsilon) \sim \Sigma\, \delta_n(\epsilon) F_n(z_c)$$
$$\sim \Sigma\, \delta_n(\epsilon)\left[F_n^{\pm}(x) \pm i\epsilon y^{\pm}(x) F_n'^{\pm}(x) + \ldots\right] \tag{4.4.6}$$

where (2.16) has been used for z_c and $F_n^{\pm}(x) = F_n(x \pm i0)$.

The system of equations (4.4.1) – (4.4.4) are now expanded inaccordance with (4.4.5) and (4.4.6). The O(1) – terms are:

$$W_{10}^{\pm}(x) + W_{10}^{\mp}(x) - f_{10}^{\mp}(x) = 0 , \tag{4.4.7}$$

$$W_{20}^{\pm}(x) + W_{20}^{\mp}(x) - f_{20}^{\mp}(x) = W_{10}^{\pm}(x) + W_{10}^{\mp}(x) - f_{10}^{\mp}(x) , \tag{4.4.8}$$

$$W_{10}^{\pm}(x) = \gamma W_{20}^{\pm}(x) + \gamma^{*}\left[W_{20}^{\pm}(x) + W_{20}^{\mp}(x) - f_{20}^{\mp}(x)\right] , \tag{4.4.9}$$

$$W_{20}(z) = Wz , \quad f_{20}(z) = fz \qquad \text{as } z \to \infty , \tag{4.4.10}$$

where $|x| < 1$. As $\epsilon \to 0$ the interface boundary is asymptotically close to the traction–free boundary. This is formally implied by (4.4.7) and (4.4.8). The vanishing of the left–hand side of (4.4.8), together with (4.4.10), leads to the ordinaary crack solution, viz.

$$f_{20}(z) = fz , \tag{4.4.11}$$

$$2W_{20}(z) - f_{20}(z) = \sigma X(z) , \tag{4.4.12}$$

where

$$X(z) = (z^2 - 1)^{1/2} . \tag{4.4.13}$$

Equations (4.4.7) and (4.4.9) are identically satisfied by

$$f_{10}(z) = \gamma f_{20}(z) = \gamma fz\ , \tag{4.4.14}$$

$$2W_{10}(z) - f_{10}(z) = \gamma\Big[2W_{20}(z) - f_{20}(z)\Big] = \gamma\sigma X(z)\ . \tag{4.4.15}$$

Unlike the situation described in chapter 3. the analysis for the present case depends heavily on the shape $z_c(x;\epsilon)$, (2.16). For example, if z_c is an ellipse then the boundary–layer variable should be scaled by ϵ^2 and the required asymptotic sequence is just $\delta_n(\epsilon)=\epsilon^n$. On the other hand, if $y^{\pm}(x)=1$ in (2.16) the required asymptotic sequence may involve $\epsilon \ell n \epsilon$. For this reason, we shall restrit our attention to the determination of a one–term approximation for a round–nosed inhomogeneity with (3.6) as the assocciated boundary–layer complex variable..

To the first order of approximation the inner expansion of the outer expansion near z = 1 is

$$f_2(1+\epsilon\zeta;\epsilon) \sim f + o(\epsilon^{1/2})\ . \tag{4.4.16}$$

$$2W_2(1+\epsilon\zeta;\epsilon) - f_2(1+\epsilon\zeta;\epsilon) \sim \epsilon^{1/2}\sqrt{2}\,\sigma\,\zeta^{1/2} + \cdots \tag{4.4.17}$$

$$f_1(1+\epsilon\zeta;\epsilon) \sim \gamma f + o(\epsilon^{1/2})\ . \tag{4.4.18}$$

$$2W_1(1+\epsilon\zeta;\epsilon) - f_1(1+\epsilon\zeta;\epsilon) \sim \epsilon^{1/2}\gamma\,\sqrt{2}\,\sigma\,\zeta^{1/2} + \cdots \tag{4.4.19}$$

4.4.4 Inner Expansion Near z = 1

For a generic function $F(z;\epsilon)$, the associated inner expansion is defined by

$$F^*(\zeta;\epsilon) \sim F_0^*(\zeta) + \epsilon^{1/2}F_1^*(\zeta) + \cdots\cdots \tag{4.4.20}$$

where $\delta_1(\epsilon) = \epsilon^{1/2}$ is determined from (4.4.16)–(4.4.19). Thus

$$f_1^*(\zeta;\epsilon) \sim f_{10}^*(\zeta) + \epsilon^{1/2} f_{11}^*(\zeta) + \cdots \tag{4.4.21}$$

$$\begin{aligned} 2W_1^*(\zeta;\epsilon) - f_1^*(\zeta;\epsilon) \sim & \left[2W_{10}^*(\zeta) - f_{10}^*(\zeta)\right] \\ & + \epsilon^{1/2}\left[2W_1^*(\zeta) - f_{11}^*(\zeta)\right] + \cdots\cdots \end{aligned} \tag{4.4.22}$$

$$f_2^*(\zeta;\epsilon) \sim f_{20}^*(\zeta) + \epsilon^{1/2} f_{21}^*(\zeta) + \cdots\cdots \tag{4.4.23}$$

$$\begin{aligned} 2W_2^*(\zeta;\epsilon) - f_2^*(\zeta;\epsilon) \sim & \left[2W_{20}^*(\zeta) - f_{20}^*(\zeta)\right] \\ & + \epsilon^{1/2}\left[2W_{21}^*(\zeta) - f_{21}^*(\zeta)\right] + \cdots\cdots \end{aligned} \tag{4.4.24}$$

In the ζ–plane, D_2 is infinite and D_1 is a semi–infinite strip Fig 1.2. Equation (4.4.1) requires $f_1^*(\zeta;\epsilon)$ to be holomorphic and $2W_1^*-f_1^*$ double valued along the negative axis. The continuity conditions are just (4.4.2) (4.4.3). Their conditions at $\zeta = \infty$ are governed by (4.4.16)–(4.4.19). The choice

$$f_{10}^*(\zeta) = \gamma f_{20}^*(\zeta) = \gamma f\,, \tag{4.4.25}$$

$$2W_{10}^*(\zeta) - f_{10}^*(\zeta) = 2W_{20}^*(\zeta) - f_{20}^*(\zeta) = 0, \tag{4.4.26}$$

meets all conditions and yields no useful information.

Before proceeding, we give a more precise description of the interface boundary $z=z_c$ in terms of the boundary–layer variable ζ defined by (3.6). It is

$$\zeta = \zeta_c = \frac{z_c - 1}{\epsilon} = \rho_c(\varphi)\, e^{i\varphi} \quad (-\pi < \varphi < \pi) \tag{4.4.27}$$

and

$$\rho_c(\varphi)\, e^{i\varphi} \to -\infty \quad \text{as} \quad \varphi \to \pm\pi\ . \tag{4.4.28}$$

Moreover, the scaling of the ϵ is chosen in such a way that

$$\text{Max}\ \rho_c(\varphi)\cos\varphi = 1\ . \tag{4.4.29}$$

It follows that

$$\left|\frac{\zeta_c}{\zeta_c - 2}\right| \leq 1\ . \tag{4.4.30}$$

Finally, the minimum value of $\rho_c(\varphi)$ is denoted by ρ_m , i.e.,

$$\rho_m \equiv \text{Min}\ \rho_c(\varphi) \leq 1\ . \tag{4.4.31}$$

The desired asymptotic limit is governed by f^*_{11}, W^*_{11}, f^*_{21} and W^*_{21}. For the region D_2 in the ζ – plane, the following series are assumed :

$$f^*_{21}(\zeta) \Big/ \frac{\sigma^*}{\sqrt{2}} = \sum_{n=1}^{\infty} A_n\, (\zeta/\rho_m)^{-n/2} \tag{4.4.32}$$

$$\left[2W^*_{21}(\zeta) - f^*_{21}(\zeta)\right] \Big/ \frac{\sigma^*}{\sqrt{2}} = 2\,\zeta^{1/2}\left\{1 + \sum_{n=1}^{\infty} B_n(\zeta/\rho_m)^{-n/2}\right\} \tag{4.4.33}$$

where σ^* is defined by

$$\sigma^* = \begin{cases} \sigma_{22} & \text{for} \quad \sigma_{12} = 0 \\ -i\sigma_{12} & \qquad\quad \sigma_{11} = \sigma_{22} = 0 \end{cases} \tag{4.4.34}$$

so that $\sigma/\sigma^* = 1$ for both cases (c.f. (2.10) for definition of σ). It is noted that the above satisfy the conditions at $\zeta = \infty$ required by (4.4.16) and (4.4.17). The form of the series follows from the analytic structure revealed by the benchmark analysis.

We employ the property (4.4.30) to set up the series solution for region D_1. It is

$$f_{11}^*(\zeta) \Big/ \frac{\sigma^*}{\sqrt{2}} = \sum_{n=0}^{\infty} a_n \left(\frac{\zeta}{\zeta - 2}\right)^n , \tag{4.4.35}$$

$$\left[2W_{11}^*(\zeta) - f_{11}^*(\zeta)\right] \Big/ \frac{\sigma^*}{2} = 2\,\gamma\,\zeta^{1/2} \sum_{n=0}^{\infty} b_n \left(\frac{\zeta}{\zeta-2}\right)^n , \tag{4.4.36}$$

where

$$\sum_{n=0}^{\infty} a_n = 0 \quad , \quad \sum_{n=0}^{\infty} b_n = 1 \ . \tag{4.4.37}$$

so that the conditions at $\zeta = \infty$, (4.4.18) and (4.4.19) , are identically satisfied.

The constants involved in the series are then determined from the interface conditions that have forms similar to (4.3.45) and (4.3.46) via the Fast Fourier Transform Algorithm. The associated stress–intensity factors are :

$$K_1^{(1)} - i\,K_2^{(1)} \equiv (\,K_I - i\,K_{II}) \Big/ \sigma_{22}\sqrt{\pi} \sim \gamma\, b_0 \ , \tag{4.4.38}$$

$$K_2^{(2)} + i\,K_1^{(2)} \equiv (\,K_{II} + i\,K_I) \Big/ \sigma_{12}\sqrt{\pi} \sim \gamma\, b_0 \ , \tag{4.4.39}$$

where b_0 is the same complex number solved from the same mathematical equations which, via the scaling factor σ^*, govern the solutions to two different physical problems.

4.4.5 <u>Results and Discussion</u>

The computer code is used to compute the K (∞, b) curve discussed in section 3.6 and presented in Fig. 3.2. Other relevant comments may be found in section 3.6.

CHAPTER V

ESHELBY TENSOR AND THE ASSOCIATED ENERGY RELEASE RATES

5.1 Plane Elasticity – The τ – Problem

We begin by reviewing the well–established results of plane elasticity. The exposition will be delineated in such a way that parallel development in the manupilation of Eshelby tensor in a homogeneous region will be demonstrated. Most of the results can be found in Eshelby's publications which will be cited at the proper places. The purpose is to summarize all the known results and the many still not fully utilized concepts advanced by Eshelby in a unified setting.

Let O – (z_1, z_2, z_3) be rectangular Cartesian coordinates a d let $(\underset{\sim}{i}_1, \underset{\sim}{i}_2, \underset{\sim}{i}_3)$ be the associated unit vectors. The two dimensional Kroncker delta and alternator are denoted by $\delta_{\alpha\beta}$ and $e_{\alpha\beta 3}$. A typical two dimensional region in (z_1, z_2) will be denoted by R, its boundary by ∂R, and the outward unit normal to ∂R by $\underset{\sim}{n}$. The displacement, strain and stress are denoted by u_α, $\epsilon_{\alpha\beta}$ and $\tau_{\alpha\beta}$, respectively. The strain–displacement and stress–strain relations are:

$$\epsilon_{\alpha\beta} = 1/2\,(u_{\alpha,\beta} + u_{\beta,\alpha}) \tag{5.1}$$

$$\tau_{\alpha\beta} = \partial f / \partial \epsilon_{\alpha\beta} = 2\,\mu\,\epsilon_{\alpha\beta} + \lambda\,\delta_{\alpha\beta}\,\epsilon_{\gamma\gamma} \tag{5.2}$$

where f is the strain–energy ensity function, and μ and λ the Lame' constants.

We shall first restrict ourselves to the case of constant μ and λ and consider the τ–problem defined by :

$$\tau_{\alpha\beta} = \tau_{\beta\alpha} \;, \tag{5.3}$$

$$\tau_{\beta\alpha,\beta} = 0 \;, \tag{5.4}$$

$$\nabla^2 \tau_{\gamma\gamma} = 0 \;, \tag{5.5}$$

$$\tau_{\beta\alpha} n_\beta = T_\alpha \;, \tag{5.6}$$

where ∇^2 is the Laplacian and T_α are prescribed on ∂R. There are the following associated relations :

$$\nabla^2 \epsilon_{\gamma\gamma} = 0 , \tag{5.7}$$

$$\nabla^2 u_\alpha = -\frac{\lambda+\mu}{\mu} \epsilon_{\gamma\gamma,\alpha} , \tag{5.8}$$

$$\omega_{,\gamma} = \frac{\lambda+2\mu}{\mu} e_{\gamma\beta 3} \epsilon_{\alpha\alpha,\beta} , \tag{5.9}$$

$$\nabla^2 \omega = 0 , \tag{5.10}$$

where

$$\omega = e_{\alpha\beta 3} u_{\alpha,\beta} . \tag{5.11}$$

Equilibrium considerations lead to the necessary conditions

$$\int_{\partial R} T_\alpha \, d\sigma = 0 , \tag{5.12}$$

$$\int_{\partial R} e_{\gamma\alpha 3} z_\gamma T_\alpha \, d\sigma = 0 , \tag{5.13}$$

and the identity

$$\int_R \tau_{\gamma\alpha} \, da = \int_{\partial R} z_\gamma T_\alpha \, d\sigma , \tag{5.14}$$

which is commonly used to define average stress in terms of boundary traction. The trace of the above may be related to the trace of

$$\int_R \epsilon_{\alpha\beta} da = 1/2 \int_{\partial R} (u_\alpha n_\beta + u_\beta n_\alpha) \, d\sigma , \tag{5.15}$$

which is commonly used to define average strain in terms of boundary displacement.

Multiplying (5.4) by u_γ and integrating over R, we obtain

$$\int_{\partial R} u_\gamma T_\alpha d\sigma = \int_R \tau_{\alpha\beta} u_{\alpha,\beta} da \quad . \tag{5.16}$$

The trace of the above is Clapeyron's Theorem :

$$\int_{\partial R} T_\alpha u_\alpha d\sigma = 2\int_R f\, da \quad . \tag{5.17}$$

Forming the skew–symmetric part of (5.16), we obtain

$$\int_{\partial R} e_{\alpha\beta 3} u_\alpha T_\beta d\sigma = \int_R e_{\alpha\beta 3} u_{\alpha,\gamma} \tau_{\alpha\beta} da \quad . \tag{5.18}$$

which must hold for all R.

Finally, in terms of the complex variable $z = z_1 + iz_2$ and two holomorphic functions $W(z)$ and $w(z)$, there are the following results (England, 1971) :

$$2\mu (u_1 + iu_2) = \kappa W - z\overline{W'} - \overline{w} \tag{5.19}$$

$$\tau_{11} + \tau_{22} = 2 (W' + \overline{W'}) \tag{5.20}$$

$$(\tau_{11} - \tau_{22}) + i2\, \tau_{12} = -2 (z\overline{W''} + \overline{w'}) \tag{5.21}$$

$$(T_1 + iT_2)\, d\sigma = -i\, d[\, W + z\overline{W'} + \overline{w}\,] \tag{5.22}$$

$$\begin{aligned}&[(z_1T_2 - z_2T_1) + i(z_1T_1 + z_2T_2)]\, d\sigma \\ &\quad = -d\left[z\,\overline{z}\,W' + zw - \int w dz - i\,\mathrm{Im} \int 2\overline{z}\,W' dz\right]\end{aligned} \tag{5.23}$$

$$\begin{aligned}&\left[(T_2u_1 - T_1u_2) + i(T_1u_1 + T_2u_2)\right] d\sigma \\ &\quad = -\frac{1}{2\mu} (\kappa W - z\overline{W'} - \overline{w})\, d(\overline{W} + \overline{z}\,W' + w)\end{aligned} \tag{5.24}$$

where

$$\kappa = \begin{cases} (3-4\nu) & \text{plane strain} \\ (3-\nu)\ /\ (1+\nu) & \text{plane stress} \end{cases} \tag{5.25}$$

and ν Poisson's ratio. The imaginary part of (5.23) and real part of (5.24) are not commonly used in studying elasticity problems. These well–known results are recorded here because, as we shall see, the Eshelby tensor permits a representation that is identical in form to (5.19)–(5.24). In particular, the full form of (5.23) and (5.24) are associated with the now well known conservation integrals.

5.2 The p – Problem and τ^* – Problem

We shall discuss in this section a mathematical problem, the p–problem, governing *the solution of a tensor field* $p_{\alpha\beta}$ *which is not symmetric. The p–problem is defined by*

$$p_{\alpha\alpha} = 0\ , \tag{5.26}$$

$$p_{\beta\alpha,\beta} = 0\ , \tag{5.27}$$

$$\nabla^2 e_{\alpha\beta 3}\, p_{\alpha\beta} = 0\ , \tag{5.28}$$

$$p_{\beta\alpha}\, n_\beta = P_\alpha\ , \tag{5.29}$$

where P_α are prescribed on ∂R. The problem is well posed, as there are four unknowns and four equations. Indeed, the situation becomes apparent if we convert the p–problem into a τ^*–problem via the transformation:

$$\tau^*_{\alpha\beta} = e_{\alpha\lambda 3}\, p_{\beta\lambda} = e_{\beta\lambda 3}\, p_{\alpha\lambda}\ , \tag{5.30}$$

$$T^*_\alpha = e_{\alpha\lambda 3}\, P_\lambda\ . \tag{5.31}$$

we shall call $\tau^*_{\alpha\beta}$ a pseudo–stress field.

Substituting the above into (5.26)–(5.29), we find that the equations governing $\tau^*_{\alpha\beta}$ are identical in form to (5.3)–(5.6). It follows that a τ^*–problem is a τ–problem. As a result, holomorphic functions $W^*(z)$ and $w^*(z)$ may be obtained to defined the pseudo–stress field $\tau^*_{\alpha\beta}$.

Since no "kinematic variables" are involved in the p–problem, and hence the τ^*–problem, we may even stretch the above mathematical analogy further by introducing pseudo strain rate $\dot{\epsilon}^*_{\alpha\beta}$ via the inverse of

$$\tau^*_{\alpha\beta} = 2\mu^* \dot{\epsilon}^*_{\alpha\beta} + \lambda^* \delta_{\alpha\beta} \dot{e}^*_{\gamma\gamma} , \tag{5.32}$$

where μ^* and λ^* are pseudo viscosity coefficients. Since the $\tau^*_{\alpha\beta}$ field is "Compatible" by (5.30) and (5.28), the psseudo strain rate $\dot{\epsilon}^*_{\alpha\beta}$ may be integrated to yield a pseudo velocity field $\dot{u}^*_\alpha$ via

$$\dot{\epsilon}^*_{\alpha\beta} = \frac{1}{2} (\dot{u}^*_{\alpha,\beta} + \dot{u}^*_{\beta,\alpha}) . \tag{5.33}$$

To obtain the solution to a mathematical p–problem we merely express the associated τ^*–problem solution in terms of $W^*(z)$ and $w^*(z)$, and then make the substitutions :

$$p_{12} - p_{21} = \tau^*_{11} + \tau^*_{22} , \tag{5.34}$$

$$(p_{12} + p_{21}) + i\, 2\, p_{22} = (\tau^*_{11} - \tau^*_{22}) + i\, 2\, \tau^*_{21} , (p_{11} = -p_{22}), \tag{5.35}$$

$$(P_1 + i\, P_2)\, d\sigma = i\, (T^*_1 + i\, T^*_2)\, d\sigma , \tag{5.36}$$

$$\begin{aligned} &\left[(z_1 P_2 - z_2 P_1) + i\,(z_1 P_1 + z_2 P_2)\right] d\sigma \\ &\quad = -i \left[(z_1 T^*_2 - z_2 T^*_1) + i\,(z_1 T^*_1 + z_2 T^*_2)\right] d\sigma. \end{aligned} \tag{5.37}$$

The asterisked versions of (5.12) and (5.13) yield

$$\int_{\partial R} P_\alpha \, d\sigma = 0 \ , \qquad \int_{\partial R} z_\alpha \, P_\alpha \, d\sigma = 0 \ , \tag{5.38}$$

and the asterisked version of the trace of (5.14) is

$$\int_{\partial R} e_{\alpha\beta 3} \, z_\alpha \, P_\beta \, d\sigma = \int_R e_{\alpha\beta 3} \, p_{\alpha\beta} \, da \ . \tag{5.39}$$

The byproduct of the solution to a τ^*–problem is a pseudo velocity field $\dot{u}^*_\alpha$ which may be expressed in terms of W^* and w^* via the assumed μ^* and λ^*. We may therefore introduce yet another pseudo velocity field v_α to pair with the tensor $p_{\alpha\beta}$. It is defined as follows :

$$v_\alpha = -\, e_{\alpha\beta 3} \, \dot{u}^*_\beta \ , \qquad \dot{u}^*_\alpha = e_{\alpha\beta 3} \, v_\beta \ . \tag{5.40}$$

5.3 Eshelby Tensor Associated with A τ – Problem

For the sake of argument and convenience, let us assume that the strain–energy density function f introduced in (5.2) depends explicitly on place and time, i.e., $f = f(\epsilon_{\alpha\beta} ; z_\gamma ; t)$. Using Eshelby's language, we may regard $f(\ldots; z_\gamma, t)$ as a calculating machine which works out the value of f when $\epsilon_{\alpha\beta}$ are inserted. The explicit dependence on z_α and t then means that the machine characteristics are continuous functions of place and time. We introduce the notation:

$$\frac{\eth f}{\eth t} = \left.\frac{\partial f}{\partial t}\right|_{\epsilon_{\alpha\beta} \text{ and } z_\alpha \text{ constant}} , \tag{5.41}$$

$$\frac{\eth f}{\eth z_\gamma} = \left.\frac{\partial f}{\partial z_\gamma}\right|_{\epsilon_{\alpha\beta, t} \text{ and } z_\alpha (\alpha \neq \gamma) \text{ constant}} , \tag{5.42}$$

so that

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \epsilon_{\alpha\beta}} \dot{\epsilon}_{\alpha\beta} + \frac{\bar{\partial} f}{\partial t} , \tag{5.43}$$

$$\frac{\partial f}{\partial z_\gamma} = \frac{\partial f}{\partial \epsilon_{\alpha\beta}} \epsilon_{\alpha\beta,\gamma} + \frac{\bar{\partial} f}{\partial z_\gamma} . \tag{5.44}$$

Let $p_{\beta\alpha}$ be the Eshelby tensor (1951, 1954, 1956, 1970). In terms of our notation the tensor is defined by

$$p_{\beta\alpha} = f\,\delta_{\beta\alpha} - \tau_{\beta\gamma}\, u_{\gamma,\alpha} . \tag{5.45}$$

Using (5.1)–(5.4) and (5.44), we may easily shows that

$$p_{\beta\alpha,\beta} = \frac{\bar{\partial} f}{\partial z_\alpha} , \tag{5.46}$$

where the right–hand side is zero if f is homogeneous in space. We now show that for such a case the Eshelby tensor field is the solution to a p – problm defined by (5.26)–(5.29). Equation (5.26) is identically satisfied by (5.45). Equation (5.27) follows from (5.46) in a homogeneous region. Using (5.1) and (5.2), we find from (5.45) that

$$e_{\alpha\beta 3}\, p_{\alpha\beta} = -(\lambda + \mu)\, \omega\, u_{\gamma,\gamma} . \tag{5.47}$$

It follows from (5.7), (5.9) and the above that (5.28) is also satisfied. The necessary conditions (5.36) are the three well–known conservation integrals (Budiansky and Rice,1973; Chadwick, 1975; Golebiewska–Herrmann. 1981,1982; Knowles and Sternberg, 1972; Rice, 1968). This completes the assertion that the Eshelby–tensor field is the solution to a p – problem. Finally, using the identity

$$e_{\alpha\beta 3}\, p_{\alpha\beta} = - e_{\alpha\beta 3}\, \tau_{\alpha\gamma}\, u_{\gamma,\beta} = - e_{\alpha\beta 3}\, \tau_{\beta\gamma}\, u_{\alpha,\gamma} \tag{5.48}$$

we obtain from (5.39) that

$$\int_{\partial R} e_{\alpha\beta 3}\, z_\alpha\, P_\beta\, d\sigma + \int_{\partial R} e_{\alpha\beta 3}\, u_\alpha\, T_\beta\, d\sigma = 0 \tag{5.49}$$

which is another one of the conservation integrals[2]

We shall say that a p – problem and a τ – problem are associated problems in a homogeneous region if $p_{\alpha\beta}$ is the Eshelby tensor associated with the τ – problem. We recall that the solution to a τ – problem may be expressed in terms of (W, w), and the solution to a p – problem may be obtained via the solutionof a τ^*– problem expressed in terms of (W^*, w^*). A tedious but straightforward calculation leads to the following simple relations for a perfect associations :

$$W^{*\prime}(z) = -\frac{\kappa+1}{4\mu}\, i\, (W'(z))^2 \,, \tag{5.50}$$

$$w^{*\prime}(z) = -\frac{\kappa+1}{4\mu}\, i\, 2\, W'(z)\, w'(z) \,. \tag{5.51}$$

5.4 Eshelby–Tensor force in Elastostatics

The many important implications of Eshelby tensor are discovered by Eshelby via the consideration of the total energy of a system. In elastostatics, the total energy E_{tot} is the potential energy. Using Eshelby's machine description f ($\epsilon_{\alpha\beta}$; z_γ, t), the total energy at time t is

$$E_{tot}(t) = \int_R f\, da - \int_{\partial R} T_\alpha\, u_\alpha\, d\sigma \tag{5.52}$$

It follows from the equilibrium conditions (5.2) and the notation (5.43) that

$$\dot{E}_{tot}(t) = \int_R \frac{\partial f}{\partial t}\, da \tag{5.53}$$

[2]The first two equations of (5.38) are commonly referred to as J_α are even more commonly referred to crack–tip contours (Rice,1968). In view of the resultant force appearance, it is perhaps more appropriate to call them F_α– integrals. The last of (5.38) is commonly called the M–integral.

which will be examined in conjunction with (5.46), i.e.,

$$p_{\beta\alpha,\beta} = \frac{\eth f}{\partial z_\alpha} \quad . \tag{5.54}$$

The discussion cannot be continued without first specifying a machine $f(\cdots; z_\gamma, t)$. The machines that have been considered by Eshelby and many other researchers are mostly piecewisely uniform in space. To fix ideas, let us introduce a curve or contour C defined by

$$C : \; z_\alpha = x_\alpha(t) \tag{5.55}$$

and hence $v_\alpha = \dot{x}_\alpha(t)$ defines the velocity of C. If the machine $f(\cdots; z_\gamma, t)$ is uniform in space on both sides of C, then for a region R containing C

$$\dot{E}_{tot}(t) = \int_R \frac{\eth f}{\partial t} da = -\int_C (f^+ - f^-) v_\alpha n_\alpha d\sigma \tag{5.56}$$

where $f^\pm$ are the uniform machines on the $\pm \underset{\sim}{n}$ sides of C. Since $f(\cdots; z_\alpha, t)$ is a step function in space, the right-hand side of (5.56) is

$$-\int_C (f^+ - f^-) v_\alpha n_\alpha d\sigma = -\int_R \frac{\eth f}{\partial z_\alpha} v_\alpha da \quad , \tag{5.57}$$

and hence

$$\dot{E}_{tot}(t) = -\int_C (P_\alpha^+ - P_\alpha^-) v_\alpha d\sigma \quad , \tag{5.58}$$

which was first obtained by Eshelby (1951) and is the essence of the physical significance of the Eshelby tensor. If C is the boundary ∂R of a region R and $v_\alpha = \dot{x}_\alpha$ is linear in x_α, (5.58) gives the energy release rates associated with the "hypothetical", in Budiansky and Rice's words (1973), translation, rotation and expansion of R. Eshelby identified (5.58) with the interface force associated with a martenstic transformation.

The various energy–release rates associated with (5.58) play the roles of generalized forces and hence are of fundamental importance in analyzing fatigue crack–damage propagation. If the crack–damage description could be approximated by the two classes of problems examined in this thesis, then analytic means must be provided to perform the nontrival calculations implied by (5.58). This was our motivation and we have completed our self–imposed goal. The following flow chart indicates the sequence of steps that may be straightforwardly performed to complete (5.58).

$$\boxed{\text{Series solution}} \rightarrow \boxed{\begin{matrix} W(z) \\ w(z) \end{matrix}} \xrightarrow[(5.44)]{\text{via } (5.50)} \boxed{\begin{matrix} W_*^*(z) \\ w^*(z) \end{matrix}} \rightarrow \boxed{\tau^*_{\alpha\beta}}$$

$$\xrightarrow[(5.37)]{(5.34)} \boxed{P_{\alpha\beta}} \rightarrow \boxed{P_{\alpha}} \rightarrow \boxed{\text{Energy Release Rates}}$$

The linking of the energy–release rates to the damage evolution kinetics is the intent of our next objective.

CHAPTER VI

SUMMARY

Our interest in toughness of composites reinforced by particles, fibers and layers, and generalized forces associated with crack–damage propagations have led us to consider the two classes of elasticity problems stipulated in Chapter I. Two computer codes, one for each of the two cases, have been developed and are ready for adaptation. The mathematical complication associated with the small geometric parameter has been effectively removed. The remaining arbitrariness associated with a general physical problem may be handled by one or several of the conventional analytic and/or numerical methods.

The computer codes effectively give the coefficients of a series solution which is valid in the neighbourhood of a crack tip. It follows that they may be manupilated to yield any and all physical quantities of significance. In particular the various Eshelby forces along the interface boundary may be easily computed by following the flow chart provided in Chapter V.

The efficiency of the two codes are illustrated and the class of problems depicted in Fig. 3.1 is discussed in section 3.6. Since we are most interested in the situation where the inhomogeneity is softer than the medium, Mode–I SIF's for five cases are summarized in Fig. 6.1, where the normalized SIF is plotted against μ_1/μ_2 for $\nu_1 = \nu_2 = 0.2$. It is seen that the effect of an inhomogeneity is strictly of a shielding nature for the range $\mu_1/\mu_2 < 1$. Moreover, the exact value of the SIF for a specific configuration falls in the rather narrow lens–shaped region bounded by the very large inhomogeneity limit, (4.2.69), and the straight line $K = \mu_1/\mu_2$ which is the exact SIF associated with a semi–infinite crack perpendicular to a two–phase boundary (Hilton and Sih, 1971). We conclude the summary with the following rather unexpected conjecture.

A conjecture: Let a crack be surrounded by a doubly symmetric inhomogeneity of

moduli μ_1 and $\nu_1 = \nu$ which, in turn, is embedded in an infinite medium of moduli μ_2 and $\nu_2 = \nu$. The inhomogeneity may consist of two disjointed inhomogeneities surrounding the tips. The normalized Mode–I SIF is a function of the two composite parameters

$$\gamma = \frac{(1+\kappa_2)\mu_1}{(1+\kappa_1)\mu_2} = \frac{\mu_1}{\mu_2} ,$$

$$\gamma^* = \frac{1}{1+\kappa_1} \left(1 - \frac{\mu_1}{\mu_2} \right) = \frac{1}{1+\kappa} \left(1 - \frac{\mu_1}{\mu_2} \right)$$

and is denoted by $K_1(\gamma,\gamma^*)$. It satisfies

$$\gamma \leq K_1 (\gamma,\gamma^*) \leq \frac{\gamma(\gamma+1-\gamma^*)}{2(\gamma+\gamma^*)(1-2\gamma^*)}$$

for $0 \leq \gamma = \mu_1/\mu_2 \leq 1$. The right–hand side of the above is deduced from (4.2.69) and equalities hold for $\gamma = \mu_1/\mu_2 = 0$ and 1.

The curves summarized in fig. 6.1 apparently satisfy the conjecture. Curves (1) and (5) are respectively the right–hand and left–hand sides of (6.1). Curve (4) is the thin confocal ellipse and (3) the circular inhomogeneity. Curve (2) corresponds to the configuration of Fig. 3.1 with a = 0.5 and b = 2.0.

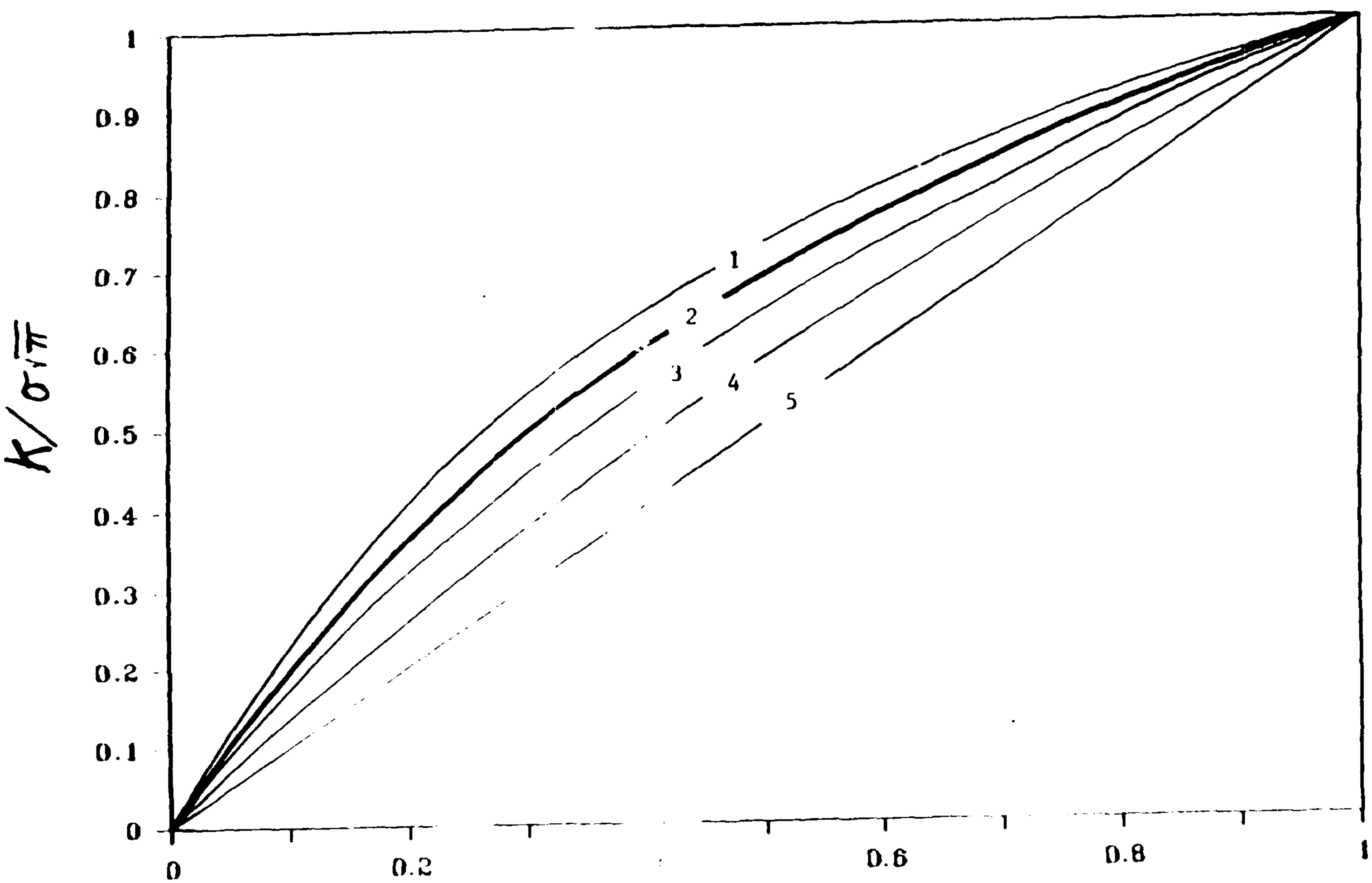


Fig. 6.1 K_1 as a ... on of μ_1/μ_2 for $\nu_1 = \nu_2 = 0.2$.
The five ... nogeneity configurations are :
1) Very ... confocal ellipse
2) The ... of Fig. 3.1
3) Crack ... ircle
4) Vani... thin confocal ellipse
5) Semi- ... crack pe... endicular to a two phase boundary.

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Appendix II

Cohesive Elasticity and Surface Phenomena

by

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To appear in Quarterly of Applied Mathematics

Cohesive Elasticity and Surface Phenomena*[1]

by

Chien H. Wu**[2]

ABSTRACT

Cohesive elasticity is the grade–3 theory of elasticity developed by Mindlin in 1965. has a modulus of cohesion which gives rise to surface–tension. The concept of adhesion is introduced, and interfacial energies and energy of adhesion are defined. The interfacial energy solution may also be used to define a grain boundary energy. Also presented are the thin film energy and the concept of an interface–phase. The stretching of a thin film is analyzed in detail; and it is found that the apparent Young's modulus obtained from a film is higher than that obtained from a plate.

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1. Introduction

Every continuum field theory contains only a limited amount of physics, and even the limited amount of physics is very often only of a phenomenological nature in the sense that the various constants employed in the theory are closely related to but not directly derived from more fundamental physical parameters. It could be argued that Young's modulus should be directly derived from more fundamental bonding forces, but the "correct" Young's modulus has always been directly measured from a tension test. It is hard to define the exact physics content of the theory of elasticity, but it is perhaps reasonable to say that most of the so-called micromechanics phenomena we are so eager to understand are not derivable from the limited amount of physics that was built in elasticity. When the implications of a continuum theory have been exhaustively revealed by the mathematic solutions to the associated field equations, additional mathematical manipulations become redundant even though new solutions will always be of usefulness.

In terms of a lattice theory, elasticity incorporates only the nearest neighbor interactions, and that is it. The theory does not have an intrinsic length scale and, as a result, a 30-cm slab behaves the same as a 40-μm film, and there is no difference between a microcrack and a geological fault. An intrinsic length scale appears when the forces between particles are extended to include first, second and n-th neighbor interactions. Moreover, an initial, homogeneous, self-equilibrating stress will lead to surface tension (Toupin and Gazis, 1964; Gazis and Willis, 1964). The surface-tension solution is of a boundary-layer type, and the associated decay constant has been estimated from electron-diffraction data obtained by Gerner, MacRae and Hartman (1961).

The continuum version of the n-th neighbor-interaction lattice theory is the so-called grade-n theory. If the strain energy-density is assumed to depend on the rotation-gradient, in addition to the strain, there results the couple stress theory. A complete grade-2 theory depends on all eighteen components of the strain-gradient, and

the additional stress quantity is sometimes termed the double stress (Toupin, 1962). The inclusion of yet a third gradient, which has thirty independent components, leads to a grade–3 theory and the associated new stress quantity is sometimes termed the triple stress (Mindlin, 1965). There is also a very general theory which includes strain–gradients of any order (Green and Rivlin, 1964).

Couple stress and higher order gradient theories were popular topics of research in the sixties, and Mindlin's work are most noteworthy in that his goal was specifically targeted at understanding the effect of microstructures on the failure of solids (Mindlin, 1962, 1964, 1965a, 1965b, 1968; Mindlin and Eshel, 1968). In particular, and to the best of our knowledge, his grade–3 theory (Mindlin, 1965) is the only continuum theory that is fully developed and has the capability of defining surface phenomena via nontrivial displacement fields. The most important concept of this theory is a new constant called the modulus of cohesion which is essentially an initial, homogeneous, self–equilibrating triple stress. It is for this most important constant that we have coined the name cohesive elasticity to stress the significance of the theory.

The availability of self–equilibrating states is fully exploited in this paper. In addition to surface free energy, which was considered by Mindlin, we have introduced the concept of an adhesive joint to define various interfacial energies and energy of adhesion. When suitably defined, an interfacial energy becomes the grain boundary energy. Also introduced are the thin film energy and the concept of an interface–phase. The stretching of a thin film is analyzed at the end. It is shown that the apparent Young's modulus obtained from a film is higher than that obtained from a slab.

The basic equations are first recapitulated in a dimensionless form. Stress function representations are then presented in terms of solutions to thirteen uncoupled second order equations. A complete boundary–layer solution is obtained for a regular boundary, and several significant, outstanding problems are outlined at the end.

2. Notation

We consider vectors in three–dimensional Euclidean space. Such vectors will be denoted by lower–case bold–face letters (**a**, **b**, **n**, etc.). Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ denote the unit vectors in the directions of the three coordinate axes z_1, z_2, z_3 of a rectangular right–handed cartesian system with origin 0. The position vector of a point P relative to 0 is denoted by $\mathbf{z}$.

In addition to scalar product $\mathbf{a}\bullet\mathbf{b}$ and vector product $\mathbf{a}\mathbf{x}\mathbf{b}$ of the two vectors **a** and **b**, the dyadic product is written $\mathbf{a}\otimes\mathbf{b}$ which, in case of no confusion, is also written **ab**. A second–order cartesian tensor **T** can be expressed as a linear combination of unit dyads, so that it takes the form

$$\mathbf{T} = T_{ij}\mathbf{e}_i\otimes\mathbf{e}_j \tag{2.1}$$

where the summation convention is employed. As a rule, we shall use bold–face capitals to denote cartesian tensors of second (and higher) order. Moreover, an over–script is used to denote the order. Thus, a cartesian tensor of order n can be expressed in components as

$$\overset{n}{\mathbf{T}} = \underbrace{T_{ij}\cdots{}_m}_{\text{n indices}}\ \underbrace{\mathbf{e}_i\otimes\mathbf{e}_j\ \otimes\cdots\otimes\ \mathbf{e}_m}_{\text{n factors}}\ . \tag{2.2}$$

The inner product $\overset{n}{\mathbf{A}}\bullet\overset{m}{\mathbf{B}}$ and outer product $\overset{n}{\mathbf{A}}\otimes\overset{m}{\mathbf{B}}$ are illustrated by

$$\overset{2}{\mathbf{A}}\bullet\overset{2}{\mathbf{B}} = A_{ij}B_{jk}\mathbf{e}_i\otimes\mathbf{e}_k\ , \tag{2.3}$$

$$\overset{2}{\mathbf{A}}\otimes\overset{2}{\mathbf{B}} = A_{ij}B_{kl}\ \mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_l\ . \tag{2.4}$$

In addition, the rule of contraction $\overset{n}{\mathbf{A}}*\overset{m}{\mathbf{B}}$ of tensors is illustrated by

$$(\mathbf{a}_1\mathbf{a}_2)*(\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3) = (\mathbf{a}_1\bullet\mathbf{b}_1)(\mathbf{a}_2\bullet\mathbf{b}_2)\mathbf{b}_3 \ , \tag{2.5}$$

$$(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3)*(\mathbf{b}_1\mathbf{b}_2) = \mathbf{a}_1(\mathbf{a}_2\bullet\mathbf{b}_1)(\mathbf{a}_3\bullet\mathbf{b}_2) \ , \tag{2.6}$$

$$(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3)*(\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3) = (\mathbf{a}_1\bullet\mathbf{b}_1)(\mathbf{a}_2\bullet\mathbf{b}_2)(\mathbf{a}_3\bullet\mathbf{b}_3) \ , \tag{2.7}$$

where, for example, $(\mathbf{a}_1\mathbf{a}_2) = \mathbf{a}_1\otimes\mathbf{a}_2$. The dyadic product indicator $\otimes$ will, for convenience, not be explicitly shown in places where no apparent confusion will be resulted from the omission.

The theory developed by Mindlin involves a material length–scale ℓ which is assumed to be much smaller than the geometric length–scale L associated with a problem. The ratio $\epsilon = \ell/L$ plays a key role in the theory, as well as in the ensuing asymptotic analyses and results. It is for this reason that we have chosen to recapitulate the theory in a dimensionless form.

Throughout this paper the length–scale L is prefixed so that, e.g.,

$$\text{Dim.}\ \mathbf{z} = L\,\mathbf{z},$$

and hence z_i are dimensionless. Let μ be the shear modulus. The force–scale used throughout this paper is just $L^2\mu$. Also, the notation Dim.( ) is used to mean dimensional form of ( ), instead of dimension of ( ).

3. Governing Equations

Let $\mathbf{u}(\mathbf{z})$ be the dimensionless displacement vector. Following the original work of Mindlin (1965), the dimensionless strain energy–density, W, is assumed to be a function of three displacement–gradient tensors:

$$W = W(\overset{2}{\mathbf{E}},\overset{3}{\mathbf{E}},\overset{4}{\mathbf{E}}) \tag{3.1}$$ [3]

where

$$\overset{2}{\mathbf{E}} = \tfrac{1}{2}\,(\nabla\mathbf{u} + \mathbf{u}\nabla),\ E_{ij} = \tfrac{1}{2}\,(u_{i,z} + u_{j,i}) \ , \tag{3.2}$$

[3]*Multiply W by μ to recover its dimension.

$$\overset{3}{\mathbf{E}} = \nabla\nabla\mathbf{u} \quad , \qquad E_{ijk} = u_{k,ij} \ , \tag{3.3}$$

$$\overset{4}{\mathbf{E}} = \nabla\nabla\nabla\mathbf{u} \quad , \qquad E_{ikjl} = u_{l,ijk} \ , \tag{3.4}$$ *4

and ∇ is the gradient operator in the dimensionless $\mathbf{z}$. The symmetry properties of the above are self–explanatory.

The variation of the total strain energy, in a (dimensionless) volume V, with arbitrary variation of $\mathbf{u}$, is

$$\delta \int_V W d\nu = \int_V [\overset{2}{\mathbf{T}}*(\nabla\delta\mathbf{u}) + \overset{3}{\mathbf{T}}*(\nabla\nabla\delta\mathbf{u}) + \overset{4}{\mathbf{T}}*(\nabla\nabla\nabla\delta\mathbf{u})]d\nu \tag{3.5}$$

where $d\nu = dz_1 dz_2 dz_3$ and

$$\overset{2}{\mathbf{T}} = \partial W/\partial\overset{2}{\mathbf{E}} \ , \ \overset{3}{\mathbf{T}} = \partial W/\partial\overset{3}{\mathbf{E}} \ , \ \overset{4}{\mathbf{T}} = \partial W/\partial\overset{4}{\mathbf{E}} \ . \tag{3.6}$$ *5

The right–hand side of (3.5) may be reduced to include a surface integral by application of the chain rule of differentiation and the divergence theorem. In doing so, however, additional caution must be taken to recognize the fact that $\nabla\delta\mathbf{u}$, together with other similar terms, is not independent of $\delta\mathbf{u}$ on the (dimensionless) surface σV of the volume V.

A number of surface operations and surface operators are needed in completing the desired reduction. Let $\mathbf{n}$ be the unit normal to ∂V and pointing away from V. The following are applicable and defined on ∂V:

$$\nabla\delta\mathbf{u} = \mathbf{n}\partial_n\delta\mathbf{u} + \nabla^o\delta\mathbf{u} \tag{3.7}$$

$$\partial_n \equiv \mathbf{n}\bullet\nabla \tag{3.8}$$

$$\nabla^o \equiv (\mathbf{I} - \mathbf{nn})\bullet\nabla \tag{3.9}$$

4*Multiply $\overset{n}{\mathbf{E}}$ by L^{-n+2} to recover its dimension.

5*Multiply $\overset{n}{\mathbf{T}}$ by μL^{n-2} to recover its dimension.

$$\nabla' \equiv \mathbf{n}(\nabla^0 \bullet \mathbf{n}) - \nabla^0 = \left[\frac{1}{R_1} + \frac{1}{R_2}\right] \mathbf{n} - \nabla^0 \tag{3.10}$$

where I is the unit tensor and $(1/R_1 + 1/R_2)$ the mean curvature of the surface. It is noted that for a flat surface ∇' is just the negative of ∇^0, the plane gradient operator.

Using the above relations, we obtain from (3.5)

$$\delta \int_V W d\nu = -\int_V \left\{\nabla\bullet \left[\overset{2}{T} - \nabla\bullet\overset{3}{T} + \nabla\bullet\left[\nabla\bullet\overset{4}{T}\right]\right]\right\} \bullet \delta u d\nu + \int_{\partial V} \left[\overset{0}{\mathbf{t}}\bullet\delta u + \overset{1}{\mathbf{t}}\bullet\partial_n\delta u + \overset{2}{\mathbf{t}}\bullet\partial_n^2 \delta u\right] da \tag{3.11}$$

where

$$\overset{0}{\mathbf{t}} = \mathbf{n} \bullet \left[\overset{2}{T} - \nabla\bullet\overset{3}{T} + (\nabla\nabla)*\overset{4}{T}\right] + \nabla'\bullet \left[\nabla'\bullet\left[\mathbf{n}\bullet\overset{4}{T}\right]\right] + \nabla'\bullet\left\{\mathbf{n}\bullet \left[\overset{3}{T}-\nabla\bullet\overset{4}{T}\right] - (\nabla^0\mathbf{n})\bullet\left[(\mathbf{nn})*\overset{4}{T}\right]\right\} , \tag{3.12}$$

$$\overset{1}{\mathbf{t}} = (\mathbf{nn})*\left[\overset{3}{T}-\nabla\bullet\overset{4}{T}\right] + \mathbf{n}\bullet\left[\nabla'\bullet\left[\mathbf{n}\bullet\overset{4}{T}\right]\right] + \nabla'\bullet\left[(\mathbf{nn})*\overset{4}{T}\right] , \tag{3.13}$$

$$\overset{2}{\mathbf{t}} = (\mathbf{nnn})*\overset{4}{T} , \tag{3.14}$$[*6]

are the generalized surface traction vectors associated with the generalized surface kinematic vectors $\mathbf{u}$, $\partial_n \mathbf{u}$ and $\partial_n^2 \mathbf{u}$, respectively. The Generalized Principal of Stationary Potential Energy and stress–equation of equilibrium are:

$$\delta \int_V W d\nu = \int_V \mathbf{f}\bullet\delta u d\nu + \int_{\partial V} \left[\overset{0}{\mathbf{t}}\bullet\delta u+\overset{1}{\mathbf{t}}\bullet\partial_n\delta u+\overset{2}{\mathbf{t}}\bullet\partial_n^2\delta u\right] da \tag{3.15}$$

$$\nabla\bullet\left[\overset{2}{T}-\nabla\bullet\overset{3}{T}+(\nabla\nabla)*\overset{4}{T}\right] + \mathbf{f} = 0 . \tag{3.16}$$

[*6] Multiply $\overset{n}{\mathbf{t}}$ by μL^n to recover its dimension.

For homogeneous and isotropic materials, the following 2nd degree polynomial W was deduced by Mindlin:

$$W = \left[E_{ij}E_{ij} + \frac{3-K}{2(\kappa-1)} E_{ii}E_{jj}\right] + \epsilon^2\beta_0 E_{iijj}$$

$$+ \epsilon^2\left[\alpha_1 E_{ijj}E_{ikk}+\alpha_2 E_{iik}E_{kjj}+\alpha_3 E_{iik}E_{jjk} + \alpha_4 E_{ijk}E_{ijk} + \alpha_5 E_{ijk}E_{kji}\right]$$

$$+ \epsilon^4\left[\beta_1 E_{iijj}E_{kkll}+\beta_2 E_{ijkk}E_{ijll}+\beta_3 E_{iijk}E_{jkll} + \beta_4 E_{iijk}E_{llkj}+\beta_5 E_{iijk}E_{lljk}+ \beta_6 E_{ijkl}E_{ijkl}\right.$$

$$\left. + \beta_7 E_{ijkl}E_{jkli}\right] + \epsilon^2\left[\gamma_1 E_{ii}E_{jjkk}+\gamma_2 E_{ij}E_{ijkk}+\gamma_3 E_{ij}E_{kkij}\right] \qquad (3.17)^{*7}$$

where $\epsilon = \ell/L$,

$$\kappa = \begin{cases} 3-4\nu & \text{(general and plane strain)} \\ \frac{3-\nu}{1+\nu} & \text{(plane stress of elasticity)}^{*8} \end{cases} \qquad (3.18)$$

and ν Poisson's ratio. All the remaining constants are expressed in dimensionless form via μ and a material length–scale ℓ, which is assumed to be small so that $\epsilon<<1$. We shall refer to ϵ as the small parameter from time to time. For example, as the small parameter tends to zero (3.17) tends to the strain energy–density of the theory of linear elasticity. All important mathematical and physical implications of the assumed W may be found in Mindlin's original work and are amplified by the explicit introduction of ℓ and ϵ. Finally, the material constants introduced by Mindlin are

$$\left\{a_n, b_0, b_n, c_n\right\} = \mu\left\{\ell^2\alpha_n, \ell^2\beta_0, \ell^4\beta_n, \ell^2\gamma_n\right\} .$$

[7*] The dimensional forms of the various terms are illustrated by, e.g., $\epsilon^2\beta_0 E_{iijj} \to (\mu\ell^2\beta_0)(E_{iijj}/L^2)$.

[8*] The meaning of plane stress in the new theory should be reexamined.

From (3.6) and (3.17) follow the constitutive equations:

$$T_{pq} = \left[2E_{pq} + \frac{3-\kappa}{\kappa-1} E_{ii}\delta_{pq}\right] + \epsilon^2\left[\gamma_1 E_{iijj}\delta_{pq} + \gamma_2 E_{pqii} + \frac{1}{2}\gamma_3(E_{iipq}+E_{iiqp})\right] , \tag{3.19}$$

$$T_{pqr} = \epsilon^2\left[\alpha_1 (E_{pii}\delta_{qr}+E_{qii}\delta_{pr}) + \frac{1}{2}\alpha_2(E_{iip}\delta_{qr}+2E_{rii}\delta_{qp}+E_{iiq}\delta_{pr}) + 2\alpha_3 E_{iir}\delta_{pq} + 2\alpha_4 E_{pqr}+\alpha_5(E_{rqp}+E_{rpq})\right] , \tag{3.20}$$

$$T_{pqrs} = \epsilon^4\Big\{\frac{2}{3}\beta_1 E_{iijj}\delta_{pqrs} + \frac{2}{3}\beta_2 E_{jkii}\delta_{jkpqrs}\,\frac{1}{6}\beta_3\left[\left[E_{iijk}+E_{iikj}\right]\delta_{jkpqrs} + 2E_{jsii}\delta_{jpqr}\right] + \frac{2}{3}\beta_4 E_{iisj}\delta_{jpqr} + \frac{2}{3}\beta_5 E_{iijs}\delta_{jpqr} + 2\beta_6 E_{pqrs} + \frac{2}{3}\beta_7(E_{qrsp} + E_{rspq} + E_{spqr})\Big\} + \epsilon^2\left[\frac{1}{3}\gamma_1 E_{ii}\delta_{pqrs}+ \frac{1}{3}\gamma_2 E_{ij}\delta_{ijpqrs}+ \frac{1}{3}\gamma_3 E_{is}\delta_{ipqr}\right]+ \epsilon^2\,\frac{1}{3}\beta_0\delta_{pqrs} , \tag{3.21}$$

wheere δ_{ij} is the Kronecker delta and

$$\delta_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} , \tag{3.22}$$

$$\delta_{ijklmn} = \delta_{ik}\delta_{jl}\delta_{mn} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jm}\delta_{kn} . \tag{3.23}$$

The symmetry of $\overset{n}{T}$ follows from that of $\overset{n}{E}$, and the very last term of (3.21) was identified by Mindlin as a cohesive force which gives rise to surface tension. As we shall see that it also gives rise to interface phase and interphase interface energies.

Let $W^*[u]$ be the total elastic energy. Then (3.17)–(3.21) lead to the conclusion that

$$W^* = \int_V W d\nu = \frac{1}{2}\epsilon^2\beta_0 \int_V \nabla^2\nabla\bullet\mathbf{u}\, d\nu + \frac{1}{2}\int_V \left[\overset{2}{\mathbf{T}}*\overset{2}{\mathbf{E}} + \overset{3}{\mathbf{T}}*\overset{3}{\mathbf{E}} + \overset{4}{\mathbf{T}}*\overset{4}{\mathbf{E}}\right] d\nu \tag{3.24}$$

where the appearance of the first integral is a result of the fact that the β_0–term in W is linear in E_{iijj}. The second volume integral may be converted into a surface integral by following the steps from (3.6)–(3.11), and the net outcome is the Generalized Clapeyron's Theorem:

$$2\int_V W d\sigma = \int_V \mathbf{f}\cdot\mathbf{u} d\sigma + \epsilon^2\beta_0 \int_V \nabla^2\nabla\cdot\mathbf{u} d\nu + \int_{\partial V}\left[\overset{0}{\mathbf{t}}\cdot\mathbf{u} + \overset{1}{\mathbf{t}}\cdot\partial_n\mathbf{u} + \overset{2}{\mathbf{t}}\cdot\partial_n^2\mathbf{u}\right]da \,. \quad (3.25)$$

Finally, the total potential energy U^* of the system is

$$U^* = W^* - \int_V \mathbf{f}\cdot\mathbf{u} d\nu - \int_{\partial V}\left[\overset{0}{\mathbf{t}}\cdot\mathbf{u}+\overset{1}{\mathbf{t}}\cdot\partial_n\mathbf{u}+\overset{2}{\mathbf{t}}\cdot\partial_n^2\mathbf{u}\right]da \,. \quad (3.26)$$

It follows from the above that if there are no external forces, i.e. if $\overset{0}{\mathbf{t}}, \overset{1}{\mathbf{t}}, \overset{2}{\mathbf{t}}$ and f are identically zero, the total potential energy is merely

$$U^* = W^* = \frac{1}{2}\epsilon^2\beta_0 \int_V \nabla^2\nabla\cdot\mathbf{u}^{(e)} d\nu = \frac{1}{2}\epsilon^2\beta_0 \int_{\partial V} \partial_n\nabla\cdot\mathbf{u}^{(e)} da \quad (3.27)$$

where $\mathbf{u} = \mathbf{u}^{(e)}$, if exists, is a self–equilibrating state. The total potential energy of the system, (3.26), is the free energy, either the Gibbs free enerrgy or the Helmholtz free energy. When a traction–free surface is the source of the disturbance, the free energy is (3.27). It follows that the dimensionless surface free energy per unit surface area F_s is given by

$$F_s = \frac{1}{2}\epsilon^2\beta_0\partial_n\nabla\cdot\mathbf{u}^{(e)} \;, \quad (3.28)$$

the dimensional form of which is termed the surface tension by Mindlin. We shall see that (3.26) can be used to define many surface related energies.

The displacement–equation of equilibrium deduced from (3.16)–(3.21) is

$$\frac{\kappa+1}{\kappa-1} D_{11}^2 D_{12}^2 \nabla\nabla\cdot\mathbf{u} - D_{21}^2 D_{22}^2 \nabla \times \nabla \times \mathbf{u} + \mathbf{f} = 0 \quad (3.29)$$

where

$$D^2_{\lambda\mu} \equiv 1 - \epsilon^2 \alpha^2_{\lambda\mu} \nabla^2 \ , \quad \lambda = 1, 2; \ \mu = 1, 2; \tag{3.30}$$

and

$$2\,\frac{\kappa+1}{\kappa-1}\,\alpha^2_{1\lambda} = \alpha - 2\gamma \pm \left[(\alpha-2\gamma)^2 - 4\beta\frac{\kappa+1}{\kappa-1}\right]^{1/2} \tag{3.31}$$

$$2\alpha^2_{2\lambda} = \alpha' - \gamma_3 \pm [(\alpha'-\gamma_3)^2 - 4\beta']^{1/2} \tag{3.32}$$

in which λ=1,2; and

$$\alpha = 2(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5) \ , \quad \alpha' = 2(\alpha_3+\alpha_4) \ ,$$

$$\beta = 2(\beta_1+\beta_2+\beta_3+\beta_4+\beta_5+\beta_6+\beta_7) \ , \quad \beta' = 2(\beta_5+\beta_6) \ , \tag{3.33}$$

$$\gamma = \gamma_1+\gamma_2+\gamma_3 \quad .$$

As the small parameter ϵ tends to zero, (3.29) reduces to the displacement–equation of equilibrium of the theory of elasticity.

As it was pointed out by Mindlin that the conditions for positive W do not include relations between α(or α') and β, γ (or β',γ_3) and hence, supply no indications of the character, real or complex, of the four constants $\alpha_{\lambda\mu}$. In all the specific solutions considered in the sequel, the $\alpha_{\lambda\mu}$ will be treated as if they were real and positive; but complex $\alpha_{\lambda\mu}$ are equally admissible. The character, real or complex, of the four constants $\alpha_{\lambda\mu}$ dictates the behavior of the field variables. On the other hand, for the most part, the final products of the problems to be analyzed are energy expressions of one form or another. These energies turn out to be indpendent of the character of $\alpha_{\lambda\mu}$. In any case the two pairs of constants (α_{11},α_{12}) and (α_{21},α_{22}) are assumed to be positive throughout this paper. In case they are actually complex, the following substitution is presumed:

$$\alpha_{12} = \overline{\alpha}_{11} \ , \ \text{Real } \alpha_{11} > 0 \ ; \tag{3.34}$$

$$\alpha_{22} = \overline{\alpha}_{21} \ , \ \text{Real } \alpha_{21} > 0 \ ; \tag{3.35}$$

which, of course, satisfy (3.31) and (3.32).

Equations (3.5), (3.11), together with the equilibrium equation (3.16), yield the identity

$$\int_V \left[\overset{2}{\mathbf{T}}{*}(\nabla\delta\mathbf{u}) + \overset{3}{\mathbf{T}}{*}(\nabla\nabla\delta\mathbf{u}) + \overset{4}{\mathbf{T}}{*}(\nabla\nabla\nabla\delta\mathbf{u})\right]d\nu$$

$$= \int_V \mathbf{f}\bullet\delta\mathbf{u}d\nu + \int_{\partial V}\left[\overset{0}{\mathbf{t}}\bullet\delta\mathbf{u}+\overset{1}{\mathbf{t}}\bullet\partial_n\delta\mathbf{u}+\overset{2}{\mathbf{t}}\bullet\partial_n^2\delta\mathbf{u}\right]da \ , \tag{3.36}$$

which may be used to establish certain useful integral identities. The result of setting $\delta\mathbf{u}=\mathbf{c}$, an arbitrary constant vector, is

$$\mathbf{c}\bullet\left[\int_V \mathbf{f}d\nu + \int_{\partial V}\overset{0}{\mathbf{t}}da\right] = 0 \ , \tag{3.37}$$

and the vanishing of the resultant force is a necessary condition for (3.16) to have a solution. The result of setting $\delta\mathbf{u}= \boldsymbol{\omega} \times \mathbf{s}$, where $\boldsymbol{\omega}$ is an arbitrary constant vector, is

$$\boldsymbol{\omega}\bullet\left[\int_V \mathbf{s}\times\mathbf{f}d\nu + \int_{\partial V}(\mathbf{s}\times\overset{0}{\mathbf{t}} + \mathbf{n}\times\overset{1}{\mathbf{t}})da\right] = 0 \ , \tag{3.38}$$

and, hence, the vanishing of the moment becomes another necessary condition. The rest of $\overset{1}{\mathbf{t}}$ and $\overset{2}{\mathbf{t}}$, i.e. $\overset{1}{\mathbf{t}}\bullet\mathbf{n}$ and the full $\overset{2}{\mathbf{t}}$, are self–equilibrating and, hence, may be arbitrarily specified.

It is possible to obtain additional integral indentities from (3.36) which may become useful in establishing the expected new Eshelby (1951, 56, 70) tensors and the associated conservation laws (Knowles and Sternberg, 1972; Budiansky and Rice, 1973), but the possibility is not pursued further in this paper.

The stress equation of equilibrium, (3.16), may be written

$$\nabla\bullet\mathbf{T}^C+\mathbf{f} = 0 \tag{3.39}$$

where

$$\mathbf{T}^C \equiv \overset{2}{\mathbf{T}}-\nabla\bullet\overset{3}{\mathbf{T}} + \nabla\bullet(\nabla\bullet\overset{4}{\mathbf{T}}) \tag{3.40}$$

may be considered as the Cauchy stress tensor with the associated Cauchy stress vector $\mathbf{t}^C$ defined by

$$\mathbf{t}^C = \mathbf{n}\bullet\mathbf{T}^C = \overset{0}{\mathbf{t}}-\nabla'\bullet\{...\} \tag{3.41}$$

where $\nabla'\bullet\{...\}$ may be identified with the terms given in (3.12). It is clear that

$$\int_V \mathrm{f}d\nu + \int_{\partial V} \mathbf{t}^C da = 0 \ , \tag{3.42}$$

$$\int_V \mathbf{s} \times \mathrm{f}d\nu + \int_{\partial V} \mathbf{s} \times \mathbf{t}^C da = 0 \ . \tag{3.43}$$

The following surface integral indentity will enable us to show that (3.37), (3.38) are equivalent to the above.

Let $\mathbf{A}$ be an aribtrary second order tensor. We have

$$\begin{aligned} \int_{\partial V} \nabla^0\bullet\mathbf{A}da &= \int_{\partial V} \nabla^0\bullet\Big\{\mathbf{n}(\mathbf{n}\bullet\mathbf{A}) + [\mathbf{A}-\mathbf{n}(\mathbf{n}\bullet\mathbf{A})]\Big\}\, da \\ &= \int_{\partial V} (\nabla^0\bullet\mathbf{n})(\mathbf{n}\bullet\mathbf{A})da + \int_{\partial V} \nabla^0\bullet[\mathbf{A}-\mathbf{n}(\mathbf{n}\bullet\mathbf{A})]\, da \end{aligned} \tag{3.44}$$

where ∇^0 is the plane gradient operator defined by (3.9). For a smooth closed surface ∂V, the last surface integral vanishes. It follows that

$$\int_{\partial V} [(\nabla^0\bullet\mathbf{n})(\mathbf{n}\bullet\mathbf{A}) - \nabla^0\bullet\mathbf{A}]da = \int_{\partial V} \nabla'\bullet\mathbf{A}da = 0, \tag{3.45}$$

which is the desired identity.

In the usual continuum theory of thermodynamics, the work term is axiomatically defined to be

$$\int_{\partial V} \mathbf{n}\bullet\mathbf{T}^C\bullet\dot{\mathbf{u}}da \tag{3.46}$$

even if $\mathbf{T}^C$ is assumed to depend on $\nabla\mathbf{u}$, $\nabla\nabla\mathbf{u}$ and $\nabla\nabla\nabla\mathbf{u}$. This led Dunn and Serrin (1984) to conclude that "a troubling aspect of all higher–grade models is that they are in gneeral incompatible with the usual continuum theory of thermodynamics". They then proceeded

to introduce the so–called interstitial working into the work integral. Mindlin's derivation, however, clearly indicates that the correct work term is the surface integral of (3.26), and an appropriately modified continuum theory would lead to thermodynamic compatibility. Similar discrepancies also show up in other higher order gradient theories (see, e.g. Gurtin, 1989).

4. Stress Functions

The solution to the displacement–equation of equilibrium was obtained by Mindlin in terms of the solutions to four uncoupled partial differential equations of order six. We give a more direct deduction and show that Mindlin's representation can be further simplified. Moreover, our deduction is constructive in that it lends itself to the actual construction of general solutions. The well–known Galerkin–Somigliana and Papkovich–Neuber representations serve as the starting point, and the component–function approach employed by Doyle (1967) is then used to reduce the system to a set of thirteen uncoupled 2nd order equations.

The displacement–equation of equilibrium (3.29) is first rewritten in the form

$$\left[\left[\frac{\kappa+1}{\kappa-1}\,\underset{11}{D^2}\,\underset{12}{D^2} - \underset{21}{D^2}\,\underset{22}{D^2}\right]\nabla\nabla\bullet\mathbf{u} + \underset{21}{D^2}\,\underset{22}{D^2}\,\nabla^2\mathbf{u}\right] + \mathbf{f} = 0 \quad . \tag{4.1}$$

In case of zero body force, the divergence of the above yields

$$\underset{11}{D^2}\,\underset{12}{D^2}\,\nabla^2\nabla\bullet\mathbf{u} = 0 \quad . \tag{4.2}$$

Applying $\underset{11}{D^2}\,\underset{12}{D^2}\,\nabla$ to (4.1) and using (4.2), we conclude that for zero body force

$$\underset{11}{D^2}\,\underset{12}{D^2}\,\underset{21}{D^2}\,\underset{22}{D^2}\,\nabla^4\mathbf{u} = 0 \tag{4.3}$$

which, for the case $\epsilon = 0$, is just the well–known result that $\mathbf{u}$ is biharmonic. We are thus led to the Generalized Galerkin–Somigliana Representation:

$$\frac{2}{\kappa+1}\,\mathbf{u} = \underset{11}{D^2}\,\underset{12}{D^2}\,\nabla^2\mathbf{G} - \left[\underset{11}{D^2}\,\underset{12}{D^2} - \frac{\kappa-1}{\kappa+1}\,\underset{21}{D^2}\,\underset{22}{D^2}\right]\nabla\nabla\bullet\mathbf{G} \quad . \tag{4.4}$$

The result of substituting (4.4) into (4.1) is

$$D^2_{11} D^2_{12} D^2_{21} D^2_{22} \nabla^4 \mathbf{G} = -\frac{2}{\kappa+1} \mathrm{f} \tag{4.5}$$

which, together with (4.4), completes the desired generalization.

Let the first term of (4.4) be denoted by $\mathbf{B}$, i.e.

$$\mathbf{B} = D^2_{11} D^2_{12} \nabla^2 \mathbf{G} \ . \tag{4.6}$$

Then, by (4.5),

$$D^2_{21} D^2_{22} \nabla^2 \mathbf{B} = -\frac{2}{\kappa+1} \mathrm{f} \ . \tag{4.7}$$

If the objective is to reexpress (4.4) in terms of $\mathbf{B}$, the solution to a lower order equation, the remaining terms in (4.4) must be appropriately modifed.

We begin by expressing the solution to (4.7) in terms of three component functions as folllows:

$$\mathbf{B} = \mathbf{B}^{(0)} + \epsilon^2 C_{21} \mathbf{B}^{(1)} + \epsilon^2 C_{22} \mathbf{B}^{(2)} \tag{4.8}$$

$$\mathbf{B}^{(0)} = D^2_{22} D^2_{21} \mathbf{B} \ , \ \mathbf{B}^{(2)} = D^2_{22} \nabla^2 \mathbf{B} \ , \ \mathbf{B}^{(2)} = D^2_{21} \nabla^2 \mathbf{B} \ , \tag{4.9}$$

$$\nabla^2 \mathbf{B}^{(0)} = -\frac{2}{\kappa+1} \mathrm{f} \tag{4.10}$$

$$D^2_{21} \mathbf{B}^{(1)} = -\frac{2}{\kappa+1} \mathrm{f} \tag{4.11}$$

$$D^2_{22} \mathbf{B}^{(2)} = -\frac{2}{\kappa+1} \mathrm{f} \tag{4.12}$$

where

$$C_{\beta 1} = -\frac{\alpha^4_{\beta 1}}{\alpha^2_{\beta 2} - \alpha^2_{\beta 1}} \ , \ C_{\beta 2} = \frac{\alpha^4_{\beta 2}}{\alpha^2_{\beta 2} - \alpha^2_{\beta 1}} \ , \tag{4.13}$$

in which $\beta = 1,2$; and C_{11}, C_{12} are defined for a similar equation to be developed in the sequel. The validity of the above decomposition may be easily verified by direct substitution (Doyle, 1967).

The divergence of (4.6) is

$$D^2_{11} D^2_{12} \nabla^2 \nabla \bullet \mathbf{G} = \nabla \bullet \mathbf{B} \tag{4.14}$$

which, together with the first of (4.9), yields

$$D^2_{11} D^2_{12} \nabla^2 (D^2_{21} D^2_{22} \nabla \bullet \mathbf{G}) = \nabla \bullet \mathbf{B}^{(0)} . \tag{4.15}$$

From the identities, which are applicable for any $\mathbf{F}$,

$$\nabla^2 \mathbf{z} \bullet \mathbf{F} = \mathbf{z} \bullet \nabla^2 \mathbf{F} + 2 \nabla \bullet \mathbf{F} \ , \tag{4.16}$$

$$D^2_{\alpha\beta} \mathbf{z} \bullet \mathbf{F} = \mathbf{z} \bullet D^2_{\alpha\beta} \mathbf{F} - 2\epsilon^2 \alpha^2_{\alpha\beta} \nabla \bullet \mathbf{F} \ , \tag{4.17}$$

The following representation is conceived:

$$D^2_{21} D^2_{22} \nabla \bullet \mathbf{G} = \frac{1}{2} \mathbf{z} \bullet \mathbf{B}^{(0)} + \frac{1}{2} B_0 \tag{4.18}$$

where B_0 is a new scalar function. Substituting the above info (4.15), we find that B_0 must satisfy

$$D^2_{11} D^2_{12} \nabla^2 B_0 = \frac{2}{\kappa+1} \left[D^2_{11} D^2_{12} \mathbf{z} \bullet \mathbf{f} - 2\epsilon^2 (\alpha^2_{11} + \alpha^2_{12} D^2_{11}) \nabla \bullet \mathbf{f} \right]$$

$$= \frac{2}{\kappa+1} \left\{ D^2_{11} D^2_{12} \left[\mathbf{z} \bullet \mathbf{f} - 2\epsilon^2 (\alpha^2_{11} + \alpha^2_{12}) \nabla \bullet \mathbf{f} \right] - 2\epsilon^4 \alpha^2_{11} C_{11} D^2_{12} \nabla^2 \nabla \bullet \mathbf{f} \right.$$

$$\left. - 2\epsilon^4 \alpha^2_{12} C_{12} D^2_{11} \nabla^2 \nabla \bullet \mathbf{f} \right\} . \tag{4.19}$$

It follows from the above that the component representation for B_0 is:

$$B_0 = B_0^{(0)*} + \epsilon^2 C_{11} B_0^{(1)} + \epsilon^2 C_{12} B_0^{(2)} \tag{4.20}$$

$$B_0^{(0)*} = D^2_{11} D^2_{12} B_0 , \ B_0^{(1)} = D^2_{12} \nabla^2 B_0 , \ B_0^{(2)} = D^2_{11} \nabla^2 B_0 \ , \tag{4.21}$$

$$\nabla^2 B_0^{(0)*} = \frac{2}{\kappa+1} \left[\mathbf{z} \bullet \mathbf{f} - 2\epsilon^2 (\alpha^2_{11} + \alpha^2_{12}) \nabla \bullet \mathbf{f} \right] \ , \tag{4.22}$$

$$D^2_{11} B_0^{(1)} = - \frac{4}{\kappa+1} \epsilon^2 \alpha^2_{11} \nabla \bullet \mathbf{f} \tag{4.23}$$

$$D^2_{12} B_0^{(2)} = - \frac{4}{\kappa+1} \epsilon^2 \alpha^2_{12} \nabla \bullet \mathbf{f} \ . \tag{4.24}$$

We have thus effectively expressed the last term of (4.4) in terms of $\mathbf{B}^{(0)}$ and B_0.

Substituting (4.8) and (4.9) into (4.14), we get

$$\nabla^2\left[D^2_{11} D^2_{12} \nabla\bullet\mathbf{G} - \epsilon^2\left[C_{21} D^2_{22} + C_{22} D^2_{21}\right]\nabla\bullet\mathbf{B}\right] = \nabla\bullet\mathbf{B}^{(0)} \tag{4.25}$$

which, in view of (4.16), permits the convenient representation

$$D^2_{11} D^2_{12} \nabla\bullet\mathbf{G} = \epsilon^2(C_{21} D^2_{22} + C_{22} D^2_{21})\nabla\bullet\mathbf{B} + \tfrac{1}{2}\,\mathbf{s}\bullet\mathbf{B}^{(0)} + \tfrac{1}{2}\,B^{(0)}_{0} \tag{4.26}$$

where $B^{(0)}_{0}$ satisfies

$$\nabla^2 B^{(0)}_{0} = -\,\mathbf{s}\bullet\nabla^2\mathbf{B}^{(0)} = \frac{2}{\kappa+1}\,\mathbf{s}\bullet\mathbf{f}\ . \tag{4.27}$$

The second term on the right–hand side of (4.4) is now expressed in terms of $\mathbf{B}$ and $B^{(0)}_{0}$ by (4.26). In view of (4.22) and (4.27), it is convenient to decompose $B^{(0)*}_{0}$ into two parts as follows:

$$B^{(0)*}_{0} = \phi + B^{(0)}_{0}, \tag{4.28}$$

$$\nabla^2\phi = -\frac{4}{\kappa+1}\,\epsilon^2(\alpha^2_{11} + \alpha^2_{12})\nabla\bullet\mathbf{f}\ . \tag{4.29}$$

The final form of the Generalized Papkovich–Neuber Representation is obtained by substituting (4.6), (4.18) and (4.26) into (4.4), viz.

$$\begin{aligned} 2\mathbf{u} = {} & (\kappa+1)\mathbf{B} - \nabla\mathbf{s}\bullet\mathbf{B}^{(0)} + \tfrac{1}{2}\,\nabla\left[(\kappa-1)B_{0} - (\kappa+1)B^{(0)}_{0}\right] \\ & - \epsilon^2(\kappa+1)(C_{21} D^2_{22} + C_{22} D^2_{21})\nabla\nabla\bullet\mathbf{B} \end{aligned} \tag{4.30}$$

where the last term may be further simplified by (4.8) – (4.13). The most explicit form of the representation is

$$\begin{aligned} 2\mathbf{u} = {} & \left[(\kappa+1)\mathbf{B}^{(0)} - \nabla\mathbf{s}\bullet\mathbf{B}^{(0)} - \nabla B^{(0)}_{0}\right] \\ & \epsilon^2(\kappa+1)\left[C_{21}\mathbf{B}^{(1)} + C_{22}\mathbf{B}^{(2)} - (\alpha^2_{21} + \alpha^2_{22})\nabla\nabla\bullet\mathbf{B}^{(0)}\right. \\ & + \tfrac{1}{2}\left[\frac{\kappa-1}{\kappa+1}\right]\nabla\left.\left[\epsilon^{-2}\phi + C_{11} B^{(1)}_{0} + C_{12} B^{(2)}_{0}\right]\right] \\ & - \epsilon^4(\kappa+1)\nabla\nabla\bullet\left[C_{21}\alpha^2_{21}\mathbf{B}^{(1)} + C_{22}\alpha^2_{22}\mathbf{B}^{(2)}\right] \end{aligned} \tag{4.31}$$

where $\epsilon^{-2}\phi$ is the particular solution of (4.29), as the homogeneous solution, which is harmonic, may be absorbed by $B_0^{(0)}$. The elasticity solution is recovered as ϵ tends to zero. We note that each one of the component functions is governed by a second order equation. The functions $B^{(0)}$, $B_0^{(0)}$ and ϕ are associated with the Laplacian operator and, hence, has no boundary–layer phenomena. The functions $B^{(\alpha)}$, $B_0^{(\alpha)}$ $(\alpha = 1,2)$ are governed by the operators $D^2_{\alpha\beta}$ defined by (3.30) and contribute to the new boundary–layer phenomena.

For antiplane deformations defined by $u_3(z_1, z_2)$ we have

$$2u_3 = (\kappa+1)B_3^{(0)} + \epsilon^2(\kappa+1)(C_{21}B_3^{(1)} + C_{22}B_3^{(2)}) \tag{4.32}$$

where $B_3^{(\alpha)}$ are of the boundary–layer type. For plane deformations defined by $u_\alpha(z_1, z_2)$ the displacements are

$$\begin{aligned} 2u_\alpha = \kappa B_\alpha^{(0)} - B_{0,\alpha}^{(0)} - z_\beta B_{\beta,\alpha}^{(0)} &+ \epsilon^2(\kappa+1)\Big[C_{21}B_\alpha^{(1)} + C_{22}B_\alpha^{(2)} - (\alpha_{21}^2 + \alpha_{22}^2)B_{\beta,\beta\alpha}^{(0)} \\ &+ \frac{1}{2}\left[\frac{\kappa-1}{\kappa+1}\right]\left[C_{11}B_{0,\alpha}^{(1)} + C_{12}B_{0,\alpha}^{(2)} \quad {}^{-2}\phi_{,\alpha}\right]\Big] \\ &- \epsilon^4(\kappa+1)\left[C_{21}\alpha_{21}^2 B_{\beta,\beta\alpha} + C_{22}\alpha_{22}^2 B_{\beta,\beta\alpha}^{(2)}\right] \end{aligned} \tag{4.33}$$

where $B_\alpha^{(\beta)}$, $B_0^{(\beta)}$ are of the boundary–layer type. It follows that the terms $B_{0,\alpha}^{(1)}$ are expected to have the most significant contribution to the new boundary–layer phenomena. Thus, the constants α_{11} and α_{12} are also expected to be more significant than α_{21} and α_{22}. We shall see that the former pair appears in all the surface energetic expressions.

5. Boundary Layer Along A Smooth Boundary

The appearance of the small parameter ϵ in the operator $D^2_{\alpha\beta}$ defined by (3.30) indicates that the main difference between elasticity and cohesive elasticity lies in a boundary layer. An elasticity solution is but the interior solution of an associated cohesive

elasticity problem. The self–equilibrating state $u = u^{(e)}$ given in (3.27) is a boundary–layer phenomenon and was discussed by Mindlin.

To fix ideas, let us consider a body with a smooth boundary defined by $\mathbf{s} = \mathbf{s}^0$. The following traction boundary–value problem is considered for the case $f \equiv 0$:

$$\overset{0}{\mathbf{t}} = \overset{0}{\tau}(\mathbf{s}^0) \tag{5.1}$$

$$\overset{1}{\mathbf{t}} = \epsilon\overset{1}{\tau}(\mathbf{s}^0;\epsilon), \tag{5.2}$$

$$\overset{2}{\mathbf{t}} = \epsilon^2\overset{2}{\tau}(\mathbf{s}^0;\epsilon), \tag{5.3}$$

where the functions on the right–hand side are prescribed generalized tractions, which may be specified to depend on ϵ. The explicit appearance of ϵ in the governing equation (3.29) and surface–traction expressions (3.12) – (3.14) suggests the appropriateness of a regular perturbation solution in the form

$$\mathbf{u}(\mathbf{s},\epsilon) \sim \mathbf{u}^{(0)}(\mathbf{s}) + \epsilon\mathbf{u}^{(1)}(\mathbf{s}) + \ldots \tag{5.4}$$

Since (3.29) contains only powers of ϵ^2, it is clear that both $u^{(0)}$ and $u^{(1)}$ satisfy the ordinary elasticity equation, i.e.

$$\text{[Elasticity Operator] } \mathbf{u}^{(0)} \text{ and } \mathbf{u}^{(1)} = 0 \ , \tag{5.5}$$

which are of second order. The solutions to (5.5) cannot be adjusted to satisfy the three sets of boundary conditions (5.1) – (5.3). The expansion (5.4) is thus termed an outer–expansion (Cole, 1968, and Van Dyke, 1975). The appropriate boundary conditions for (5.5) can only be determined from matching. To facilitate such a computation it is necessary to set up a set of normal coordinates relative to the boundary surface $\mathbf{s} = \mathbf{s}^0$. Such coordinate systems are routinely used in the theory of shells.

Let ζ_α $(\alpha=1,2)$ denote a set of Gaussian coordinates, so that the boundary surface is given by

$$\mathbf{s} = \mathbf{s}^0 = \mathbf{s}^0(\zeta_\alpha) = \mathbf{s}^0(\zeta_1,\zeta_2) \ . \tag{5.6}$$

The associated base vectors and first fundamental tensor are denoted by $\mathbf{a}_\alpha$ and $a_{\alpha\beta}$.

In addition, a third base vector a_3 is identified with the normal $\mathbf{n}$ to the surface, so that the position vector $\mathbf{x}$ becomes

$$\mathbf{x} = \mathbf{x}^0(\xi_\alpha) + \xi_3 \mathrm{a}_3 \ , \quad \mathrm{a}_3 = \mathbf{n} \tag{5.7}$$

where ξ_3 is the dimensionless distance from the surface along $\mathbf{n}$. The interior of the body is therefore defined by $\xi_3 < 0$.

In terms of the normal coordinates a_i, the displacement vector is

$$\mathbf{u}(\mathbf{x};\epsilon) = \overset{*}{\mathbf{u}}(\xi) = \overset{*}{u}_i(\xi)\ \mathbf{a}_1/\sqrt{\mathrm{a}_{ii}} \ , \tag{5.8}$$

so that $\overset{*}{u}_i$ are the physical components. The outer–expansion (5.4) is now rewritten as

$$\mathbf{u} \sim \left[\mathrm{u}_j^{*(0)}(\xi_i) + \epsilon\, \mathrm{u}_j^{*(0)}(\xi_\alpha,0)\xi_3 + \cdots \right] \mathbf{a}_j/\sqrt{\mathrm{a}_{jj}} \tag{5.9}$$

which, for small values of ξ_3, may be approximated by

$$\begin{aligned} \mathbf{u} \sim \Big\{ &\left[\mathrm{u}_j^{*(0)}(\xi_\alpha,0) + \mathrm{u}_{j,3}^{*(0)}(\xi_\alpha, 0)\xi_3 + \cdots \right] \\ &+ \epsilon \left[\mathrm{u}_j^{*(1)}(\xi_\alpha, 0) + \mathrm{u}_{j,3}^{*(1)}(\xi_\alpha,0)\xi_3 + \cdots \right] + \cdots \Big\} \mathbf{a}_j/\sqrt{\mathrm{a}_{jj}} . \end{aligned} \tag{5.10}$$

The coordinate ξ_3 is now stretched to become a boundary–layer coordinate ζ defined by

$$\zeta = \xi_3/\epsilon \ . \tag{5.11}$$

The inner–expansion of the outer–expansion is obtained by substituting (5.11) into (5.10), and the result is

$$\mathbf{u} \sim \left\{ \mathrm{u}_j^{*(0)}(\xi_\alpha,0) + \epsilon \left[\mathrm{u}_j^{*(1)}(\xi_\alpha,0) + \zeta \mathrm{u}_{j,3}^{*(0)}(\xi_\alpha,0) \right] + \cdots \right\} \mathbf{a}_j/\sqrt{\mathrm{a}_{jj}} . \tag{5.12}$$

Let $\tau_{ij}^{*(n)}(\xi_i)$, $n = 0,1$, denote the physical components of the ordinary elasticity stress tensors associated with the ordinary elasticity solution $\mathrm{u}_i^{*(n)}(\xi_i)$, i.e.

$$\overset{*}{\tau}_{ij} = 2\overset{*}{\epsilon}_{ij} + \frac{3-\kappa}{\kappa-1}\,\delta_{ij}\overset{*}{\epsilon}_{kk} \tag{5.13}$$

where $\overset{*}{\epsilon}_{ij}$ are the physical components of the ordinary strain tensor. On the surface $\xi_3 = 0$, we have

$$\epsilon^*_{33} = u^*_{3,3}(\xi_\alpha, 0)$$

$$\epsilon^*_{\alpha\alpha} = \frac{1}{\sqrt{a_{\alpha\alpha}}} u^*_{\alpha,\alpha}(\xi_\beta, 0) + \frac{1}{R_{\alpha\alpha}} u^*_3(\xi_\beta, 0) \quad (\alpha = 1, 2; \text{ no sum}) \tag{5.14}$$

$$\epsilon^*_{\alpha 3} = \frac{1}{2}\left[\frac{1}{\sqrt{a_{\alpha\alpha}}} u^*_{3,\alpha}(\xi_\beta, 0) + u^*_{\alpha,3}(\xi_\beta, 0) - \frac{1}{R_{\alpha\alpha}} u^*_\alpha(\xi_\beta, 0)\right]$$

where R_{11}, R_{22} are the normal radii of curvature. The curvatures are positive if the centers of curvature are on the side $\xi_3 < 0$.

The desired inner—expansion, consistent with (5.12), is

$$\mathbf{u} \sim \left\{u^{*(0)}_j(\xi_\alpha, 0) + \epsilon v_j(\xi_\alpha, \xi) + \dots\right\} \mathbf{a}_j / \sqrt{a_{jj}} \tag{5.15}$$

where v_j are again physical components. The governing equations deduced from (3.29) are

$$\left[1 - \alpha^2_{11} \frac{\partial^2}{\partial \xi^2}\right]\left[1 - \alpha^2_{12} \frac{\partial^2}{\partial \xi^2}\right] \frac{\partial^2 v_3}{\partial \xi^2} = 0 \ , \tag{5.16}$$

$$\left[1 - \alpha^2_{21} \frac{\partial^2}{\partial \xi^2}\right]\left[1 - \alpha^2_{22} \frac{\partial^2}{\partial \xi^2}\right] \frac{\partial^2 v_\alpha}{\partial \xi^2} = 0 \ . \tag{5.17}$$

The leading terms of the displacement gradient tensors are:

$$\begin{gathered} E^*_{33} = v_{3,\xi} \ , \ E^*_{3\alpha} = \frac{1}{2}\left[v_{\alpha,\xi} + 2\epsilon^{*(0)}_{\alpha 3} - u^{*(0)}_{\alpha,3}\right] \ , \\ E^*_{\alpha\beta} = \epsilon^{*(0)}_{\alpha\beta} \ ; \\ E^*_{33i} = \frac{1}{\epsilon} v_{i,\xi\xi} \ ; \\ E^*_{333i} = \frac{1}{\epsilon^2} v_{i,\xi\xi\xi} \ ; \end{gathered} \tag{5.18}$$

where all components are physical and refer to the base $\mathbf{a}_i$.

There are more than a few nonvanishing stress components associated with the above but only the following are needed for the boundary conditions:

$$T^*_{33} = \frac{\kappa+1}{\kappa-1} v_{3,\xi} + \gamma v_{3,\xi\xi\xi} + \frac{3-\kappa}{\kappa-1} \epsilon^{*(0)}_{\alpha\alpha} \ ,$$

$$T^*_{3\alpha} = T^*_{\alpha 3} = v_{\alpha,\xi} + \frac{\gamma}{2} v_{\alpha,\xi\xi\xi} + 2\epsilon^{*(0)}_{\alpha 3} - u^{*(0)}_{\alpha,3} \ ;$$

$$T^*_{333} = \epsilon\alpha v_{3,\xi\xi} \ , \tag{5.19}$$

$$T^*_{33\beta} = \epsilon 2(\alpha_3 + \alpha_4) v_{\beta,\xi\xi} \ ;$$

$$T^*_{3333} = \epsilon^2 \left[(\beta_0 + v_{3,\xi} + \beta v_{3,\xi\xi\xi}) + \gamma_1 \epsilon^{*(0)}_{\alpha\alpha}\right]$$

$$T^*_{333\alpha} = \epsilon^2\left\{\left[\frac{1}{2}\gamma_3 v_{\alpha,\xi} + 2(\beta_5 + \beta_6) v_{\alpha,\xi\xi\xi}\right] + \gamma_3 (2\epsilon^{*(0)}_{\alpha 3} - u^{*(0)}_{\alpha,3})\right\}$$

Finally, the leading terms of the traction vectors are obtained from (3.12) – (3.14). They are

$$\overset{0}{\mathbf{t}} = \left[T^*_{3i} - \frac{1}{\epsilon} T^*_{33i,\xi} + \frac{1}{\epsilon^2} T^*_{333i,\xi\xi}\right] a_i/\sqrt{a_{ii}} \ ,$$

$$\overset{1}{\mathbf{t}} = \left[T^*_{33i} - \frac{1}{\epsilon} T^*_{333i,\xi}\right] a_i/\sqrt{a_{ii}} \ , \tag{5.20}$$

$$\overset{2}{\mathbf{t}} = T^*_{333i}\, a/\sqrt{a_{ii}} \ .$$

The traction boundary conditions (5.1) – (5.3) are first rewritten in terms of ξ and a_i, i.e.

$$\overset{0}{\mathbf{t}} = \overset{0}{\tau}{}^*(\xi_\alpha),\ \overset{1}{\mathbf{t}} = \epsilon\overset{1}{\tau}{}^*(\xi_\alpha,\epsilon),\ \overset{2}{\mathbf{t}} = \epsilon^2\overset{2}{\tau}{}^*(\xi_\alpha;\epsilon) \ . \tag{5.21}$$

The final form of the boundary conditions are

$$\beta_0 + \gamma v_{3,\xi} + \beta v_{3,\xi\xi\xi} = \overset{2}{\tau}{}^*_3(\xi_\alpha;0) - \gamma_1 \epsilon^{*(0)}_{\alpha\alpha} \tag{5.22}$$

$$(\alpha-\gamma)\, v_{3,\xi\xi} - \beta v_{3,\xi\xi\xi\xi} = \overset{1}{\tau}{}^*_3(\xi_\alpha;0) \ , \tag{5.23}$$

$$\frac{\kappa+1}{\kappa-1} v_{3,\xi} + (2\gamma-\alpha) v_{3,\xi\xi\xi} + \beta v_{3,\xi\xi\xi\xi\xi} = \overset{0}{\tau}{}^*(\xi_\alpha) - \frac{3-\kappa}{\kappa-1}\epsilon^{*(0)}_{\alpha\alpha}(\xi_\alpha,0); \tag{5.24}$$

$$\frac{1}{2}\gamma_3 v_{\lambda,\xi} + 2(\beta_5+\beta_6) v_{\lambda,\xi\xi\xi} = \overset{2}{\tau}{}^*(\xi_\beta,0) - \gamma_3(2\epsilon^{*(0)}_{\lambda 3} - u^{*(0)}_{\lambda,3}) \ , \tag{5.25}$$

$$\left[2(\alpha_3 + \alpha_4) - \tfrac{1}{2}\gamma_3\right] v_{\lambda,\zeta\zeta} - 2(\beta_5 + \beta_6) v_{\lambda,\zeta\zeta\zeta\zeta} = \overset{1*}{\tau}_{\lambda}(\zeta_\beta;0) \ , \tag{5.26}$$

$$v_{\lambda,\zeta} + \left[\gamma_3 - 2(\alpha_3 + \alpha_4\right] v_{\lambda,\zeta\zeta\zeta} + 2(\beta_5 + B_6) v_{\lambda,\zeta\zeta\zeta\zeta\zeta\zeta}$$

$$= \overset{0*}{\tau}_{\lambda}(\zeta_\beta) + u^{*(0)}_{\alpha,3}(\zeta_\beta,0) - 2\epsilon^{*(0)}_{\alpha 3}(\zeta_\beta,0) \tag{5.27}$$

The following identities, which are obtained from (3.31), will be used repeatedly:

$$\alpha^2_{11} + \alpha^2_{12} = \frac{\kappa-1}{\kappa+1}(\alpha - 2\gamma) \ , \ \alpha^2_{11}\alpha^2_{12} = \frac{\kappa-1}{\kappa+1}\beta \ ;$$

$$\alpha^2_{21} + \alpha^2_{22} = 2(\alpha_3 + \alpha_4) - \gamma_3 \ , \ \alpha^2_{21}\alpha^2_{22} = 2(\beta_5 + \beta_6) \ . \tag{5.28}$$

Using these identities, we find that (5.24), (5.27) are, respectively, the first integrals of the governing equations (5.16), (5.17). The solutions to these equations, for $\zeta < 0$, are

$$v_3 = u^{*(1)}_3(\zeta_\alpha,0) + \zeta u^{*(0)}_{3,3}(\zeta_\alpha,0) + A_{13} e^{\zeta/\alpha_{11}} + B_{13} e^{\zeta/\alpha_{12}} \ , \tag{5.25}$$

$$v_\beta = u^{*(1)}_\beta(\zeta_\alpha,0) + \zeta u^{*(0)}_{\beta,3}(\zeta_\alpha,0) + A_{2\beta} e^{\zeta/\alpha_{21}} + B_{2\beta} e^{\zeta/\alpha_{22}} \ ; \tag{5.26}$$

and

$$u^{*(0)}_{3,3}(\zeta_\alpha,0) = \frac{\kappa-1}{\kappa+1}\left[\overset{0}{\tau}{}^*_3(\zeta_\alpha) - \frac{3-\kappa}{\kappa-1}\epsilon^{*(0)}_{\alpha\alpha}(\zeta_\alpha,0)\right] \ , \tag{5.27}$$

$$u^{*(0)}_{\beta,3}(\zeta_\alpha,0) = \overset{0*}{\tau}_\beta(\zeta_\alpha) + u^{*(0)}_{\beta,3}(\zeta_\alpha,0) - 2\epsilon^{*(0)}_{\beta 3}(\zeta_\alpha,0) \ ; \tag{5.28}$$

where the last two conditions are the results of matching, i.e. the outer–expansion of the inner–expansion must match the inner–expansion of the outer expansion given by (5.12). Using (5.13) and (5.15), we find from the above that the boundary conditions for the elasticity solution $u^{*(1)}_i(\zeta_j)$ are just the traction conditions:

$$\tau^{*(0)}_{i3}(\zeta_\alpha,0) = \overset{0*}{\tau}_i(\zeta_\alpha) \ , \tag{5.29}$$

and, hence, the first term of the outer–expansion (5.9) is completely defined.

The remaining boundary conditions (5.22), (5.23) and (5.25), (5.26) involve only derivatives of $\mathbf{v}$ and, hence, the constant terms $u^{*(1)}_i(\zeta_\alpha,0)$ cannot be determined from the inner expansion. A lengthy calculation yields the following concise results for the other

two sets of constants:

$$A_{\beta j} = \frac{\left[\alpha^2_{\beta 0} + \alpha^2_{\beta_2}\right] \alpha^2_{\beta_1} L_{j2} - \left[\alpha^2_{\beta 0} + \alpha^2_{\beta_1}\right] \alpha_{\beta_1} L_{j1}}{\left[\alpha^2_{\beta 0} + \alpha^2_{\beta_2}\right]^2 \alpha_{\beta_1} - \left[\alpha^2_{\beta 0} + \alpha^2_{\beta_1}\right]^2 \alpha_{\beta_2}} , \tag{5.30}$$

$$B_{\beta j} = \frac{-\left[\alpha^2_{\beta_0} + \alpha^2_{\beta_1}\right] \alpha^2_{\beta_2} L_{j2} + \left[\alpha^2_{\beta_0} + \alpha^2_{\beta_2}\right] \alpha_{\beta_2} L_{j1}}{\left[\alpha^2_{\beta_0} + \alpha^2_{\beta_2}\right]^2 \alpha_{\beta_1} - \left[\alpha^2_{\beta_0} + \alpha^2_{\beta_1}\right]^2 \alpha_{\beta_2}} , \tag{5.31}$$

where

$$\alpha^2_{10} = \frac{\kappa-1}{\kappa+1} \gamma \quad , \quad \alpha^2_{20} = \frac{1}{2} \gamma_3 \; , \tag{5.32}$$

$$L_{31} = \alpha_{11} \alpha_{12} \frac{\kappa-1}{\kappa+1} \overset{1}{\tau}{}^*_3 (\xi_\alpha;0) \; , \tag{5.33}$$

$$L_{32} = -\frac{\kappa-1}{\kappa+1} \beta_0 - \alpha^2_{10} u^{*(0)}_{3,3} (\xi_\alpha,0) + \frac{\kappa-1}{\kappa+1} \left[\overset{2}{\tau}{}^*_3 (\xi_\alpha,0) - \gamma_1 \epsilon^{*(0)}_{\alpha\alpha}\right] \tag{5.34}$$

$$L_{\beta_1} = \alpha_{21} \alpha_{22} \overset{1*}{\tau}_\beta (\xi_\alpha;0) \; , \tag{5.35}$$

$$L_{\beta_2} = -\alpha^2_{20} u^{*(0)}_{\beta,3} (\xi_\alpha,0) + \overset{2*}{\tau}_\beta (\xi_\alpha;0) - \gamma_3 (2\epsilon^{*(0)}_{\beta 3} - u^{*(0)}_{\beta,3}) . \tag{5.36}$$

With the exception of the terms $u^{*(1)}_i (\xi_\alpha,0)$, which are but constants in the inner–expansion (5.15), the desired two–term inner–expansion, (5.15), is now complete.

Had we actually began with the presumption that $\alpha_{\lambda\mu}$ are complex, (3.34) and (3.35), then $B_{\beta j}$ should be replaced by $\bar{A}_{\beta j}$ and (5.30) and (5.31) still hold, as the latter is merely the complex conjugate of the former.

6. Traction–Free Surface Free Engery

Consider now a body, of volume V and bounding surface ∂V, that is completely free of external forces, i.e.

$$f = \overset{0}{\tau}{}^*(\xi_\alpha) = \overset{1}{\tau}{}^*(\xi_\alpha;\epsilon) = \overset{2}{\tau}{}^*(\xi_\alpha;\epsilon) = 0 \quad . \tag{6.1}$$

It follows from (5.229) that the first term of the outer–expansion, $u^{*(0)}(s)$, is identically zero and, hence,

$$L_{32} = -\frac{\kappa-1}{\kappa+1}\beta_0 \ , \ L_{31} = L_{\beta_1} = L_{\beta_2} = 0 \ , \tag{6.2}$$

by (5.33) – (5.36). The inner–expansion (5.5) is merely

$$u = u^{(e)} \sim \epsilon v_3(\xi)\ a_3/\sqrt{a_{33}} \ , \tag{6.3}$$

$$v_3 = A_{13} e^{\xi/\alpha_{11}} + B_{13} e^{\xi/\alpha_{12}} \ , \tag{6.4}$$

where

$$A_{13} = -\frac{\kappa-1}{\kappa+1}\alpha_{11}^2(\alpha_{10}^2 + \alpha_{12}^2)\,\beta_0/D_1 \ , \tag{6.5}$$

$$B_{13} = \frac{\kappa-1}{\kappa+1}\alpha_{12}^2(\alpha_{10}^2 + \alpha_{11}^2)\,\beta_0/D_1 \ , \tag{6.6}$$

$$D_1 = (\alpha_{10}^2 + \alpha_{12}^2)^2\,\alpha_{11} - (\alpha_{10}^2 + \alpha_{11}^2)^2\,\alpha_{12} \ , \tag{6.7}$$

which are determined by (5.30), (5.31).

The above solution is a self–equilibrating state and, hence, (3.27) and (3.28) apply. Thus,

$$U^* = U_e^* = AF_s \ , \ F_s = \frac{1}{2}\epsilon\frac{\kappa-1}{\kappa+1}\beta_0^2(\alpha_{11}^2 - \alpha_{12}^2)/D_1 \tag{6.8}$$

where A is the total surface area and F_s the free surface energy associated with the traction–free surface of a completely unloaded body. The dimensional form of F_s is

$$\text{Dim. } F_s = \frac{(\kappa-1)\ b_0^2\ (\alpha_{11}^2 - \alpha_{12}^2)}{2(\kappa+1)\mu\ell^3\ D_1}$$

$$= \frac{(\kappa-1)b_0^2}{2(\kappa+1)\mu\ell^3}\ \frac{\alpha_{11} + \alpha_{12}}{(\alpha_{10}^2 - \alpha_{11}\alpha_{12})^2 - \alpha_{11}\alpha_{12}(\alpha_{11} + \alpha_{12})^2} \ , \tag{6.9}$$

which was first obtained by Mindln. Thus, to the first order of magnitude, as $\epsilon \to 0$, the

surface energy associated with a regular traction–free boundary of a completely unloaded body is a material constant.

The expression (6.8) remains the same if α_{12} is the complex conjugate of α_{11}, (3.34). Since F_S must be positive, we have

$$(\alpha_{10}^2 - \alpha_{11}\alpha_{12})^2 - \alpha_{11}\alpha_{12}(\alpha_{11} + \alpha_{12})^2 > 0 \;, \tag{6.10}$$

a condition that must be satisfied by the new material constants.

The surface energy given by (6.8) is the energy, per unit area, associated with the formation of a new **regular surface**. The free end of a thin wire is not a regular surface as a whole unless it is sufficiently thick, in terms of the material scale ℓ, as it is interacting with the free cylindrical surface. As to how thin is thin or how thick is thick the self–equilibrating state associated with a free cylinder must by analyzed. It goes without saying that the surface in the neighborhood of a notch (or crack) tip is not regular in the sense that (6.8) is also not applicable. The conclusion is that the energy required in the formation of new surfaces is not always determined by (6.8), and many relevant self–equilibrating states must be analyzed to understand the phenomenon. In short, the energy required to scoop a smooth marble out of a chunk of solid, or slice it into halfs, may be computed from (6.8). The so–called toughness may not be directly obtained from the material constant F_S.

Let us now consider a traction–free surface on a loaded body. To avoid the unnecessary complication of dividing up the bounding surface into different portions, we assume that the load is merely a body force so that the bounding surface is still completely free. The outer–expansion $u^{*(0)}(\xi)$, (5.9), is now no longer zero. Using the fact that ∂V is traction free, we have

$$u^{*(0)}_{3,3}(\xi_\alpha, 0) = -\frac{3-\kappa}{2(7-\kappa)}\,\tau^{*(0)}_{\beta\beta}(\xi_\alpha, 0) \tag{6.11}$$

where $\tau^{*(0)}_{\alpha\alpha}(\xi_\alpha, 0)$ is the surface (elasticity) stress invariant produced by the nonzero body force f. The inner–expansion associated with the still traction–free ∂V is now given

by (5.15), (5.25), (5.26), (5.30), (5.31) with

$$L_{31} = 0, \; L_{32} = -\frac{\kappa-1}{\kappa+1}\beta_0 + \alpha_{10}^2 \frac{3-\kappa}{2(7-\kappa)} \tau^{*(0)}_{\alpha\alpha}(\zeta_\alpha, 0) \; ; \tag{6.12}$$

$$L_{\beta_1} = 0, \; L_{\beta_2} = -\alpha_{20}^2 u^{*(0)}_{\beta,3}(\zeta_\alpha, 0) \; . \tag{6.13}$$

The total potential energy of the system may be computed from (3.26), (3.25), viz.

$$U^* = -\frac{1}{2}\int_V \mathbf{f}\bullet\mathbf{u}d\nu + \frac{1}{2}\epsilon^2\beta_0 \int_{\partial V} \partial_n \nabla\bullet\mathbf{u}da \; , \tag{6.14}$$

as the surface ∂V is still free. For $\epsilon = 0$, we recover the ordinary elasticity potential energy associated with the system, i.e.

$$U^*\Big|_{\epsilon=0} = \overset{*}{U}_{E1} = -\frac{1}{2}\int_V \mathbf{f}\bullet\mathbf{u}^{(0)}d\nu \tag{6.15}$$

where $\mathbf{u}^{(0)}(\mathbf{x}) = \mathbf{u}^{*(0)}(\zeta)$.

The second integral of (6.14) is now computed to give

$$U^* = \overset{*}{U}_{E1} + AF_s(\tau^*_{\alpha\alpha}) \tag{6.16}$$

where, to the first order of magnitude,

$$F_s(\tau^*_{\alpha\alpha}) = \frac{1}{2}\epsilon\beta_0 \; v_{3,33}$$

$$= \frac{1}{2}\epsilon\beta_0 \left[\frac{\kappa-1}{\kappa+1}\beta_0 - \alpha_{10}^2 \frac{3-\kappa}{2(7-\kappa)} \tau^{*(0)}_{\alpha\alpha}(\zeta_\alpha, 0)\right] (\alpha_{11}^2 - \alpha_{12}^2)/D_1 \; , \tag{6.17}$$

which is to be compared with (6.8). It is noted that the contributions of $\mathbf{u}^{(0)}$ and v_α in the surface integral of (6.14) are of one order lower than that of v_3 and, hence, do not appear in (6.17). The contribution of the modulus of cohesion β_0 is a constant triple stress, the last term of (3.21), which gives rise to $F_s(0) = F_s$ of (6.8). It follows that the interaction between β_0 and f, which gives rise to $\tau^{*(0)}_{\alpha\alpha}$, is

$$\Delta F_s = F_s(\tau^{*(0)}_{\alpha\alpha}) - F_s = -\frac{(3-\kappa)\gamma}{2(7-\kappa)\beta_0} \tau^{*(0)}_{\alpha\alpha} F_s \tag{6.18}$$

which is negative if the surface stress (not to be confused with surface tension, a term that

is deliberately avoided in our presentation) invariant $\tau^{*(0)}_{\alpha\alpha}$ is positive. The total potential energy of the system now becomes

$$U^* = U^*_{E1} + AF_s + A(\Delta F_s) \ . \tag{6.19}$$

Consider now an infinite slab of thickness H with a cylindrical hole of radius R. The slab is uniformly stretched at infinity by a tensile stress $t = \sigma e_r$ while e_r is the unit radial vector. The elasticity potential energy is

$$U^*_{E1} = U^*_\infty - \frac{\pi(\kappa+1)}{4}\sigma^2 R^2 H \tag{6.20}$$

where U^*_∞ is the energy associated with the slab without the hole and the second term is the so-called flaw (hole) energy (Sih and Liebowitz, 1967). The cohesive elasticity potential energy may be obtained from (6.16) and (6.17) with

$$\tau^{*(0)}_{\alpha\alpha} = 2\sigma + 2\nu\sigma = \frac{7-\kappa}{2}\sigma \text{ (plane strain)} \ . \tag{6.21}$$ [*9]

Thus,

$$U^* = U^*_{E1} + 2\pi RH\left[1 - \frac{(3-\kappa)\gamma}{4\beta_0}\sigma\right] F_s \ , \tag{6.23}$$

and the change of U^* associated with a hypothetical hole expansion is

$$\frac{\partial U^*}{\partial R} = -\frac{\pi(\kappa+1)}{2} HR\sigma^2 + 2\pi H\left[1 - \frac{(3-\kappa)\gamma}{4\beta_0}\sigma\right] F_s \tag{6.24}$$

where the first term is the elasticity potential energy release rate. The vanishing of (6.24) is the condition for the hypothetical expansion and the elasticity counterpart of such a condition is the Griffith criterion:

$$-\frac{\pi(\kappa+1)}{2} HR\sigma^2 + 2\pi HF_s = 0 \ . \tag{6.25}$$

The relevancy or irrelevancy of the hypothetical expansion condition is not the purpose of our discussion. Rather, we use (6.24), (6.25) to illustrate the main difference between cohesive elasticity and elast :y. The surface energy term is a part of the total potential e. ergy computed from the fie.d solution in (6.24), whereas it is merely an appended side

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[*9] The stress concentration factor is 2.

condition in (6.25). Moreover, the role of the interaction energy, (6.18), is apparent in (6.24).

7. Interfacial Energies

Let us now assume that the upper and lower half spaces $V^{\pm}$ are taken from two different solids. We propose to glue them together, by a layer of adhesive of vanishing thickness, to form a composite body V. Such a process cannot be nontrivially defined by the ordinary theory of elasticity. Noting that the most striking difference between elasticity and cohesive elasticity is the existence of a constant triple stress $T_{iijj} = \epsilon^2\beta_o$, (3.21), we assume that the desired adhesion is the result of a sigular variation in β_o. Moreover, the two half spaces $V^{\pm}$ are assumed to be identical in every respect other than the possibility of a difference in β_o. Before proceeding, it is convenient to summarize the relevant one–dimensional theory as follows:

$$\left[1 - \alpha_{11}^2 \frac{d^2}{d\xi^2}\right]\left[1 - \alpha_{12}^2 \frac{d^2}{d\xi^2}\right]\frac{d^2 v}{d\xi^2} + \frac{\kappa-1}{\kappa+1}\frac{d^3\beta_o}{d\xi^3} = 0 \ , \tag{7.1}$$

$$\overset{0}{\tau}_3 = \frac{\kappa+1}{\kappa-1}\left[\left[1 - \alpha_{11}^2 \frac{d^2}{d\xi^2}\right]\left[1 - \alpha_{12}^2 \frac{d^2}{d\xi^2}\right]\frac{dv}{d\xi}\right] \ , \tag{7.2}$$

$$\overset{1}{\tau}_3 = (\alpha-\gamma)\frac{d^2 v}{d\xi^2} - \beta\frac{d^4 v}{d\xi^4} \ , \tag{7.3}$$

$$\overset{2}{\tau}_3 = \beta_o + \gamma\frac{dv}{d\xi} + \beta\frac{d^3 v}{d\xi^3} \ , \tag{7.4}$$

where $v(\xi)$ is the v_3 of (5.16), (5.22) – (5.24), and the possibility of a spatially varying β_o is retained in the equation of equilibrium (7.1). The above equations are to be applied to $V^{\pm}$ defined by $\xi \gtrless 0$ in which the modulus of cohesion β_o is assumed to be of different constant values $\beta_o^{\pm}$. In addition, β_o is assumed to have certain singular behaviors at $\xi = 0$. Specifically, we assume that

$$\beta_o(\xi) = \beta_o^- + (\beta_o^+ - \beta_o^-)\,\delta_o(\xi) + \beta_o^*\,\delta(\xi) \ , \tag{7.5}$$

where $\delta_{o}(\xi)$ and $\delta(\xi)$ are, respectively, the unit step function and delta function, such that

$$\frac{d}{d\xi}\,\delta_{o}(\xi) = \delta(\xi), \quad \int_{-\infty}^{+\infty} \delta(\xi)\,d\xi = 1 \ . \tag{7.6}$$

The constant β^{*}_{o} is a new parameter characterizing the adhesion. For convenience, it is defined in terms of yet another parameter β^{A}_{o}, the modulus of cohesion of the adhesive material, by the expression

$$\beta^{*}_{o} = \frac{\left[\alpha_{11} + \alpha_{12}\right]\alpha^{2}_{11}\alpha^{2}_{12}}{\left[\alpha^{2}_{10} - \alpha_{11}\alpha_{12}\right]^{2} - \alpha_{11}\alpha_{12}\left[\alpha_{11} + \alpha^{2}_{12}\right]}\,\beta^{A}_{o} \tag{7.7}$$

where the dimension of the dimensional form of the factor, which is introduced for convenience (c.f. (6.9), (6.10), (3.34)), is ℓ and

$$\text{Dim. } \beta^{A}_{o} = b^{A}_{o} = \mu\ell^{2}\beta^{A}_{o} \ , \tag{7.8}$$

so that the interpretation of (7.5) is dimensionally correct.

The solution satisfying (7.1) and $\overset{0}{\tau}{}^{\pm}_{3}(0) = 0$ is

$$v^{\pm}_{,\xi}(\xi) = A^{\pm}\, e^{\mp\xi/\alpha_{11}} + B^{\pm}\, e^{\mp\xi/\alpha_{12}} \quad (\xi \gtrless 0) \tag{7.9}$$

where the four constants $A^{\pm}$, $B^{\pm}$ must now be determined to meet the implication of (7.5). It follows from all the above relations and additional continuity requirements that

$$v^{+}_{,\xi\xi\xi}(0) - v^{-}_{,\xi\xi\xi}(0) = -\frac{1}{\beta}(\beta^{+}_{o} - \beta^{-}_{o}) \ , \tag{7.10}$$

$$v^{+}_{,\xi\xi}(0) - v^{-}_{,\xi\xi}(0) = -\frac{1}{\beta}\beta^{*}_{o} \ , \tag{7.11}$$

$$v^{+}_{,\xi}(0) = v^{-}_{,\xi}(0) \ , \tag{7.12}$$

$$\overset{1}{\tau}{}^{+}_{3}(0) = \overset{1}{\tau}{}^{-}_{3}(0) \ , \tag{7.13}$$

$$\overset{2}{\tau}{}^{+}_{3}(0) = \overset{2}{\tau}{}^{-}_{3}(0) \quad \text{(identity)}, \tag{7.14}$$

where the last condition is identically satisfied if (7.10) and (7.12) are met. It is also clear

from (7.13) and (7.2) that, in addition to (7.10) and (7.11), the 4th and 5th derivatives of v are also discontinuous, even though the generalized tractions are all continuous. The condition (7.12) conforms with the ordinary elasticity requirement, as there is no concentrated force in the problem.

The four constants obtained from the application of (7.10) – (7.13) are

$$A^{\pm} = \frac{(\kappa-1)}{2(\kappa+1)} - \left[\frac{\left[\alpha_{10}^{2} + \alpha_{12}^{2}\right]\alpha_{11}}{D_1} \beta_o^A \mp \frac{\beta_o^{+} - \beta_o^{-}}{\alpha_{12}^{2} - \alpha_{11}^{2}} \right] , \tag{7.15}$$

$$B^{\pm} = \frac{(\kappa-1)}{2(\kappa+1)} \left[\frac{\left[\alpha_{10}^{2} + \alpha_{11}^{2}\right]\alpha_{12}}{D_1} \beta_o^A \mp \frac{\beta_o^{+} - \beta_o^{-}}{\alpha_{12}^{2} - \alpha_{11}^{2}} \right] , \tag{7.16}$$

where D_1 is given by (6.7). The combination of (7.9) is a self–equilibrating state, as far as the composite body V is concerned, and (3.26) gives the potential energy for the system

$$U^{*} = A\Gamma_I , \tag{7.17}$$

$$\Gamma_I = \epsilon \frac{(\kappa-1)}{4(\kappa+1)} \left[\frac{\left[\alpha_{11}^{2} - \alpha_{12}^{2}\right]\left[\beta_o^{+} + \beta_o^{-}\right]\beta_o^A}{D_1} - \frac{\left[\beta_o^{+} - \beta_o^{-}\right]^{2}}{\alpha_{11}\alpha_{12}(\alpha_{11} + \alpha_{12})} \right] , \tag{7.18}$$

where Γ_I is the interfacial free energy. Since the interfacial free energy must be nonnegative, we have

$$\beta_o^A \geq \beta_{om}^A \equiv \left[\frac{\left[\alpha_{10}^{2} - \alpha_{11}\alpha_{12}\right]^{2}}{\left[\alpha_{11} + \alpha_{12}\right]^{2}\alpha_{11}\alpha_{12}} - 1 \right] \frac{\left[\beta_o^{+} - \beta_o^{-}\right]^{2}}{\beta_o^{+} + \beta_o^{-}} \tag{7.19}$$

where the right–hand side is positive by (6.10).

The energy of adhesion, E_{Ad}, is defined as the amount of work necessary to increase the separation of two surfaces from zero to infinite distance, and is defined by (Murr, 1975)

$$E_{Ad} = F_s^+ + F_s^- - \Gamma_I \tag{7.20}$$

where $F_s^{\pm}$ are computed by (6.8). The energy of adhesion must also be non–negative. Thus,

$$\beta_o^A \leq \beta_{oM}^A \equiv \beta_o^+ + \beta_o^- + \frac{\left[\alpha_{10}^2 - \alpha_{11}\alpha_{12}\right]^2}{\left[\alpha_{11} + \alpha_{12}\right]^2 \alpha_{11}\alpha_{12}} - \frac{\left[\beta_o^+ - \beta_o^-\right]^2}{\left[\beta_o^+ + \beta_o^-\right]} . \tag{7.21}$$

In summary, we have

$$\beta_{om}^A \leq \beta_o^A \leq \beta_{oM}^A ; \tag{7.22}$$

<u>Perfect Adhesion</u>

$$\beta_o^A = \beta_{om}^A , \Gamma_I = 0 , E_{Ad} = F_s^+ + F_s^- ; \tag{7.23}$$

<u>Adhesionless Adhesion</u>

$$\beta_o^A = \beta_{oM}^A , \Gamma_I = F_s^+ + F_s , E_{Ad} = 0 . \tag{7.24}$$

For the case of identical $V^{\pm}$, i.e. $\beta_o^+ = \beta_o^- = \beta_o$, we have

$$0 = \beta_{om}^A \leq \beta_o^A \leq \beta_{oM}^A = 2\beta_o . \tag{7.25}$$

It is seen from (7.9), (7.15) and (7.16) that

$$v_{,\zeta} \equiv 0 \text{ for } \beta_o^+ = \beta_o^- = \beta_o , \beta_o^A = 0 ; \tag{7.26}$$

$$v_{,\zeta} = v_{3,\zeta} \text{ of (6.4), for } \beta_o^+ = \beta_o^- = \beta_o , \beta_o^A = 2\beta_o . \tag{7.27}$$

The last condition indicates that the traction–free solution is recovered via the singularity representation (7.5) without the specific stipulation that the generalized tractions $\overset{1}{\tau}_3$ and $\overset{2}{\tau}_3$ be zero. In this connection, it is more convenient to interpret the singular β_o–variation of (7.1) as the body force, i.e.

$$f(\xi) = \frac{d^3\beta_o}{d\xi^3} . \tag{7.28}$$

Thus, the solution to two disjointed half spaces, in the sense of (7.27), may be obtained from the solution to a full space subjected to a suitably defined singular body force. This observation opens up the possibility of defining a microcrack via the introduction of singular body force systems. A crack is termed a microcrack if its dimensionless length is of the order of ϵ and, as a result, every term of the governing equation (4.1) is of the same order, as the independent variable s must be rescaled by ϵ. The explicit component–function representation obtained in Section 4, together with the above body–force observation, lead us to believe that an explicit solution for the microcrack problem is an achievable goal. This problem is being pursued by us.

In terms of cohesive elasticity, which is still a crude continuum model for solids, a grain boundary or an interphase interface is but a special adhesive boundary. Thus, different physical interpretations could be assigned to the quantities β_o^A, Γ_I and E_{Ad}. When a grain boundary meets a free surface, a thermal groove profile for grain boundary–surface equilibrium may be induced. If the angle sustained by the groove is Ω_s, then the grain boundary free energy Γ_{gb}, which may be identified with Γ_I of (7.8), is

$$\Gamma_{gb} = 2F_s \text{Cos}\,(\Omega_s/2) . \tag{7.29}$$

The groove configuration is a singular boundary layer phenomenon, in .erms of our analytical treatement, and must be analyzed separately. It is now also possible to refine the mathematical definition of an interfacial crack – another class of singular problems to be solved.

8. Thin Film Energy and The Concept of An Interface–Phase

A layer of solid is called a plate if its thickness is much larger than the material

length scale ℓ. It is termed a film if its dimensionless thickness is of the order of ϵ. The governing equations for a free film are just (7.1) – (7.4), and the boundary conditions are

$$\overset{0}{\tau}_3 = \overset{1}{\tau}_3 = \overset{2}{\tau}_3 = 0 \ \text{ at } \ \zeta = \pm\, h \tag{8.1}$$

where 2h is the dimensionless thickness. The solution is

$$v(\zeta) = \frac{(\kappa-1)\beta_0}{(\kappa+1)D_1(h)}\left[-\alpha_{11}^2\left[\alpha_{10}^2+\alpha_{12}^2\right]\operatorname{Sinh}\frac{h}{\alpha_{12}}\operatorname{Sinh}\frac{\zeta}{\alpha_{11}}\right.$$
$$\left.+\,\alpha_{12}^2\left[\alpha_{10}^2+\alpha_{11}^2\right]\operatorname{Sinh}\frac{h}{\alpha_{11}}\operatorname{Sinh}\frac{\zeta}{\alpha_{12}}\right], \tag{8.2}$$

where

$$D_1(h) = \alpha_{11}\left[\alpha_{10}^2+\alpha_{12}^2\right]^2\operatorname{Cosh}\frac{h}{\alpha_{11}}\operatorname{Sinh}\frac{h}{\alpha_{12}}$$
$$-\,\alpha_{12}\left[\alpha_{10}^2+\alpha_{11}^2\right]^2\operatorname{Cosh}\frac{h}{\alpha_{12}}\operatorname{Sinh}\frac{h}{\alpha_{11}}, \tag{8.3}$$

which is to be compared with D_1 of (6.3) – (6.7).

The above solution is self–equilibrating, and (3.27) still applies. The total potential energy for the system is

$$U^* = AF_F, \tag{8.2}$$

$$F_F = \epsilon\,\frac{\kappa-1}{\kappa+1}\,\beta_0^2\left[\alpha_{11}^2-\alpha_{12}^2\right]\operatorname{Sinh}\frac{h}{\alpha_{11}}\operatorname{Sinh}\frac{h}{\alpha_{12}}\Big/D_1(h), \tag{8.3}$$

where F_F is the film free energy, per unit area of the film, which is to be compared with (6.8). We have

$$\lim_{h\to\infty} F_F(h) = 2F_s \tag{8.4}$$

where F_s is the free–surface free energy (6.8). The limit simply states the fact that the free surfaces of a (thick) plate do not interact. As h tends to zero, we have

$$\lim_{h\to\infty} F_F(h) = \epsilon\,\frac{\kappa-1}{\kappa+1}\,\beta_0^2\,h\Big/\left[\alpha_{10}^4-\alpha_{11}^2\alpha_{12}^2\right], \tag{8.5}$$

which is to be interpreted as the mathematical approximation for $F_F(h)$, for small values

of h. Since F_F must be non-negative, we have (c.f. (3.34))

$$\alpha_{10}^2 > \alpha_{11}\alpha_{12} . \tag{8.6}$$

It follows from the above that (6.10) may be replaced by

$$\alpha_{10}^2 - \alpha_{11}\alpha_{12} > \left[\alpha_{11} + \alpha_{12}\right]\left[\alpha_{11}\alpha_{12}\right]^{1/2} , \tag{8.7}$$

Here we see that material-parameter properties are not solely governed by the positive definiteness of the strain energy density (c.f. the paragraph before (3.34)). It is anticipated that many, many more conditions of the kind will eventually come out of many different physical situations. It can be easily checked that F_F is a monotonically increasing function of h.

Gluing two half spaces and a layer together, the layer in between is termed an interface-phase if its dimensionless thickness is of the order of ϵ. Let us assume that the two half spaces are identical, so that only half of the symmetric problem needs to be analyzed. The set of equations (7.1) – (7.4) still applies, and the lower half of the problem is defined by

$$v(\xi) = \begin{cases} v^I(\xi) \\ v(\xi) \end{cases} , \quad \beta_o(\xi) = \begin{cases} \beta_o + \frac{1}{2}\beta_o^I\left[\left[\frac{\xi}{h}\right]^2 - 1\right] & (-h<\xi<0) \\ \beta_o & (\xi<-h) \end{cases} \tag{8.8}$$

We still keep the simple assumption that all the other material properties are the same for both the half spaces and the layer.

The associated self-equilibrating solution is

$$v^I,_\xi(\xi) = \frac{A}{\alpha_{11}}\text{Cosh}\frac{\xi}{\alpha_{11}} + \frac{B}{\alpha_{12}}\text{Cosh}\frac{\xi}{\alpha_{12}} + \frac{\kappa-1}{\kappa+1}\beta_o^I/h^2 \quad (-h<\xi<0) \tag{8.9}$$

$$v,_\xi(\xi) = \frac{A_o}{\alpha_{11}}e^{(\xi+h)/\alpha_{11}} + \frac{B_o}{\alpha_{12}}e^{(\xi+h)/\alpha_{12}} \quad (\xi<-h) \tag{8.10}$$

where

$$A_o = -A\,\text{Sinh}\frac{h}{\alpha_{11}} , \quad B_o = -B\,\text{Sinh}\frac{h}{\alpha_{12}} , \tag{8.11}$$

$$A = A^*/(\mathrm{Cosh}\frac{h}{\alpha_{11}} + \mathrm{Sinh}\frac{h}{\alpha_{11}}) \,, \; B = B^*/(\mathrm{Cosh}\frac{h}{\alpha_{12}} + \mathrm{Sinh}\frac{h}{\alpha_{12}}), \tag{8.12}$$

$$A^* = \frac{\alpha_{11}^3}{\alpha_{12}^2 - \alpha_{11}^2}\frac{\kappa-1}{\kappa+1}\frac{\beta_o^I}{h^2} \,, \; B^* = -\frac{\alpha_{12}^3}{\alpha_{12}^2 - \alpha_{11}^2}\frac{\kappa-1}{\kappa+1}\frac{\beta_o^I}{h^2} \,. \tag{8.13}$$

The total potential energy of the system is again given by (3.27) except that $\beta_o(\zeta)$ is now the function defined by (8.9). We have

$$U^* = A\Gamma_{II} \,, \tag{8.14}$$

$$\Gamma_{II} = \epsilon\left[\frac{\kappa-1}{\kappa+1}\right]\frac{\left[\beta_o^I\right]^2}{h^4\left[\alpha_{12}^2 - \alpha_{11}^2\right]}$$

$$\times\left\{\frac{\alpha_{12}^2\left[h\mathrm{Cosh}\frac{h}{\alpha_{12}} - \alpha_{12}\mathrm{Sinh}\frac{h}{\alpha_{11}}\right]}{\mathrm{Sinh}\frac{h}{\alpha_{12}} + \mathrm{Cosh}\frac{h}{\alpha_{12}}} - \frac{\alpha_{11}^2\left[h\mathrm{Cosh}\frac{h}{\alpha_{11}} - \alpha_{11}\mathrm{Sinh}\frac{h}{\alpha_{11}}\right]}{\mathrm{Sinh}\frac{h}{\alpha_{11}} + \mathrm{Cosh}\frac{h}{\alpha_{11}}}\right\}, \tag{8.15}$$

where Γ_{II} is the interface–phase free energy, per unit area of the interface, which is to be compared with Γ_I of (7.18) where the interface–phase was taken as a idealized dividing surface. We have

$$\lim_{h\to 0}\Gamma_{II}(h) = \epsilon\left[\frac{\kappa-1}{\kappa+1}\right]\left[\beta_o^I\right]^2/3\alpha_{11}\alpha_{12}(\alpha_{11} + \alpha_{12}) \tag{8.16}$$

which is to be taken as the mathematic limit of Γ_{II}, for small values of h. It is noted from (8.15) and (8.16) that Γ_{II} is always positive, regardless of the sign in front of β_o^I in (8.8). In other words, Γ_{II} remains the same for

$$\beta_o(\zeta) = \beta_o - \tfrac{1}{2}\beta_o^I\left[\left[\frac{\zeta}{h}\right]^2 - 1\right] \qquad (-h<\zeta<0) \,. \tag{8.17}$$

The difference between the above and (8.9) lies in the fact that

$$\beta_o(\zeta) < \beta_o \ , \ \beta_o(0) = \beta_o - \tfrac{1}{2}\beta_o^I \ \text{ for (8.9)}$$

$$\beta_o(\zeta) > \beta_o \ , \ \beta_o(0) = \beta_o + \tfrac{1}{2}\beta_o^I \ \text{ for (8.17)} \tag{8.18}$$

The strength of such a joint is dictated by the minimum value of $\beta_o(\zeta)$.

The $\beta_o(\zeta)$ variation defined by (8.8) is arbitrary. It is used to illustrate the meaning of an interface–phase. However, it is clear that $\beta_o(\zeta)$ may be required to yield a minimum for Γ_{II}. This proposition is not pursued further in this paper. It is also apparent that the energy associated with a film deposited on top of a substrate can also be meaningfully defined.

9. Stretching of Thin Films.

Consider now a (thin) film occupying the region

$$\left|z_3\right| < \epsilon h \ \text{ or } \ \left|\zeta\right| < h \tag{9.1}$$

where $\zeta = z_3/\epsilon$. The film is biaxially stretched, so that

$$u_1 = \epsilon_{11} z_1 \ , \ u_2 = \epsilon_{22} z_2 \ , \ u_3 = \epsilon v(\zeta) \tag{9.2}$$

where ϵ_{11} and ϵ_{22} are the constant elasticity strains. For convenience, we define τ_{11} and τ_{22} by

$$\tau_{11} = \frac{\kappa+1}{\kappa-1}\epsilon_{11} + \frac{3-\kappa}{\kappa-1}\epsilon_{22} \ ,$$

$$\tau_{22} = \frac{\kappa+1}{\kappa-1}\epsilon_{22} + \frac{3-\kappa}{\kappa-1}\epsilon_{11} \ , \tag{9.3}$$

which are obtained by substituting ϵ_{11}, ϵ_{22}, $\epsilon_{33} = 0$ into the ordinary stress–strain relation with κ being the first expression (not the plane–stress version) of (3.18).

In terms of cohesive elasticity, the nonzero components of the strains are

$$E_{11} = \epsilon_{11} \ , \ E_{22} = \epsilon_{22} \ , \ E_{33} = v,_{\zeta} \ ;$$

$$E_{333} = \frac{1}{\epsilon} v,_{\zeta\zeta} \ ; \ E_{333} = \frac{1}{\epsilon^2} v,_{\zeta\zeta\zeta} \ . \tag{9.4}$$

The nonzero components of the stresses are

$$T_{\alpha\alpha} = \tau_{\alpha\alpha} + \frac{3-\kappa}{\kappa-1} v,_{\xi} + \gamma_1 v,_{\xi\xi\xi} \quad (\alpha=1,2; \text{ no sum}) , \tag{9.5}$$

$$T_{33} = \frac{3-\kappa}{\kappa-1}(\epsilon_{11} + \epsilon_{22}) + \frac{\kappa+1}{\kappa-1} v,_{\xi} + \gamma v,_{\xi\xi\xi} ; \tag{9.6}$$

$$T_{333} = \epsilon\alpha v,_{\xi\xi} , \tag{9.7}$$

$$T_{311} = T_{131} = T_{322} = T_{232} = \epsilon(\alpha_1 + \tfrac{1}{2}\alpha_2) v,_{\xi\xi} ; \tag{9.8}$$

$$T_{3333} = \epsilon^2[(\beta_0 + \gamma v,_{\xi} + \beta v,_{\xi\xi\xi}) + \gamma_1(\epsilon_{11} + \epsilon_{22})] , \tag{9.9}$$

$$\begin{aligned} T_{\alpha\beta 33} &= \epsilon^2\Big\{ \tfrac{1}{3}[\gamma_1(\epsilon_{11} + \epsilon_{22})\delta_{\alpha\beta} + \gamma_2\epsilon_{\alpha\beta}] \\ &+ \tfrac{1}{3}[\beta_0 + (\gamma_1 + \gamma_3)v,_{\xi} + (2\beta_1 + \beta_3 + 2\beta_4 + 2\beta_5) v,_{\xi\xi\xi}]\,\delta_{\alpha\beta}\Big\} , \end{aligned} \tag{9.10}$$

$$\begin{aligned} T_{33\alpha\beta} &= \epsilon^2\Big\{\tfrac{1}{3}[\gamma_1(\epsilon_{11} + \epsilon_{22})\delta_{\alpha\beta} + \gamma_3\epsilon_{\alpha\beta}] \\ &+ \tfrac{1}{3}[\beta_0 + (\gamma_1 + \gamma_2) v,_{\xi} + (2\beta_1 + 2\beta_2 + \beta_3)v,_{\xi\xi\xi}]\,\delta_{\alpha\beta}\Big\} , \end{aligned} \tag{9.11}$$

$$\begin{aligned} T_{\alpha\beta\gamma\delta} &= \epsilon^2\Big\{\tfrac{1}{3}\gamma_1(\epsilon_{11} + \epsilon_{22})\delta_{\alpha\beta\gamma\delta} + \tfrac{1}{3}\gamma_2(\epsilon_{\alpha\beta}\delta_{\gamma\delta} + \epsilon_{\alpha\gamma}\delta_{\beta\delta} + \epsilon_{\beta\gamma}\delta_{\alpha\delta}) \\ &+ \tfrac{1}{3}\gamma_3(\delta_{\alpha\beta}\epsilon_{\gamma\delta} + \delta_{\alpha\gamma}\epsilon_{\beta\delta} + \delta_{\beta\gamma}\epsilon_{\alpha\delta}) \\ &+ \tfrac{1}{3}[\beta_0 + \gamma_1 v,_{\xi} + 2\beta_1 v,_{\xi\xi\xi}]\delta_{\alpha\beta\gamma\delta}\Big\} . \end{aligned} \tag{9.12}$$

The governing equation for v is just (7.1), and the traction–free conditions on $\xi = \pm h$ are

$$\frac{\kappa+1}{\kappa-1}\left[(1 - \alpha_{11}^2 \frac{d^2}{d\xi^2})(1 - \alpha_{12}^2 \frac{d^2}{d\xi^2}) \frac{dv}{d\xi}\right] + \frac{3-\kappa}{\kappa-1}\epsilon_{\alpha\alpha} = 0 , \tag{9.13}$$

$$(\alpha-\gamma)\frac{d^2v}{d\xi^2} - \beta\frac{d^4v}{d\xi^4} = 0 , \tag{9.14}$$

$$\gamma_1\epsilon_{\alpha\alpha} + \beta_0 + \gamma\frac{dv}{d\xi} + \beta\frac{d^3v}{d\xi^3} = 0 . \tag{9.15}$$

The solution is

$$v = -\frac{3-\kappa}{\kappa+1}\epsilon_{\alpha\alpha}\zeta + A\,\mathrm{Sinh}\,\frac{\zeta}{\alpha_{11}} + B\,\mathrm{Sinh}\,\frac{\zeta}{\alpha_{12}}\ , \tag{9.16}$$

where

$$A = -\frac{\kappa-1}{\kappa+1}\alpha_{11}^2(\alpha_{10}^2+\alpha_{12}^2)\,\mathrm{Sinh}\,\frac{h}{\alpha_{12}}\left[\beta_0 + (\gamma_1 - \frac{3-\kappa}{\kappa+1}\gamma)\epsilon_{\alpha\alpha}\right]/D_1(h), \tag{9.17}$$

$$B = \frac{\kappa-1}{\kappa+1}\alpha_{12}^2(\alpha_{10}^2+\alpha_{11}^2)\,\mathrm{Sinh}\,\frac{h}{\alpha_{11}}\left[\beta_0 + (\gamma_1 - \frac{3-\kappa}{\kappa+1}\gamma)\epsilon_{\alpha\alpha}\right]/D_1(h), \tag{9.18}$$

in which $D_1(h)$ is just the expression given by (8.3). The solution reduces to (8.2), which was obtained for a film with a fixed edge, when $\epsilon_{\alpha\alpha} = 0$.

On the cross–section z_1 = constant, the stress–traction is

$$\overset{0}{t} = t_{11}e_1\ ,\ t_{11}(\zeta) = T_{11} - 2T_{311,3} + 3T_{3311,33} \tag{9.19}$$

which follows from (3.12). It is noted that the stress–traction is no longer independent of the thickness coordinate. We have

$$t_{11}(\zeta) = \tau_{11} + \tau(\zeta)\ , \tag{9.20}$$

$$\tau(\zeta) = \frac{3-\kappa}{\kappa-1}v_{,\zeta} + (2\gamma_1+\gamma_2-2\alpha_1-\alpha_2)v_{,\zeta\zeta\zeta} + (2\beta_1+\beta_2+\beta_3)v_{,\zeta\zeta\zeta\zeta\zeta}. \tag{9.21}$$

Similarly, the stress–traction on the cross–section z_2 = constant is

$$\overset{0}{t} = t_{22}e_2\ ,\ t_{22}(\zeta) = T_{22} - 2T_{322,3} + 3T_{3322,33}\ , \tag{9.22}$$

$$t_{22}(\zeta) = \tau_{22} + \tau(\zeta)\ . \tag{9.23}$$

The average of $\tau(\zeta)$ over the thickness is

$$<\tau> = \frac{1}{2h}\int_{-h}^{h}\tau(\zeta)d\zeta$$

$$= \frac{1}{h}\left[\frac{3-\kappa}{\kappa-1}v(h) + (2\gamma_1+\gamma_2-2\alpha_1-\alpha_2)\frac{d^2v(h)}{d\zeta^2} + (2\beta_1+2\beta_2+\beta_3)\frac{d^4v(h)}{d\zeta^4}\right]$$

$$= -\frac{(3-\kappa)^2}{\kappa^2-1}\epsilon_{\alpha\alpha} - \left[\beta_0 + (\gamma_1 - \frac{3-\kappa}{\kappa+1}\gamma)\epsilon_{\alpha\alpha}\right]F(h) \tag{9.24}$$

where

$$F(h) = \frac{\kappa-1}{\kappa+1}\left[\frac{3-\kappa}{\kappa+1}\gamma - (2\gamma_1+\gamma_2-2\alpha_1-\alpha_2) - (2\beta_1+2\beta_2+\beta_3)(\alpha-\gamma)/\beta\right]$$

$$\times\,(\alpha_{11}^2-\alpha_{12}^2)\,\mathrm{Sinh}\,\frac{h}{\alpha_{11}}\,\mathrm{Sinh}\,\frac{h}{\alpha_{12}}\,/\,hD_1(h)\ . \tag{9.25}$$

The averaged forms of (9.20) and (9.23) are

$$\langle t_{11}\rangle = \tau_{11} + \langle\tau\rangle\ ,\ \langle t_{22}\rangle = \tau_{22} + \langle\tau\rangle\ . \tag{9.26}$$

A necessary condition for a film of finite size to be completely free is $\langle t_{11}\rangle = \langle t_{22}\rangle = 0$. It follows that there exists an ϵ^0 such that

$$\epsilon_{11} = \epsilon_{22} = \epsilon^0\ ,\ \tau_{11} = \tau_{22} = \frac{4}{\kappa-1}\,\epsilon^0\ , \tag{9.27}$$

$$\frac{2(7-\kappa)}{\kappa+1}\,\epsilon^0 - \left[\beta_0 + (\gamma_1 - \frac{3-\kappa}{\kappa+1}\,\gamma)\ 2\epsilon^0\right] F(h) = 0\ , \tag{9.28}$$

where the last condition is merely the requirement that $\langle t_{11}\rangle = \langle t_{22}\rangle = 0$. The associated solution is therefore self–equilibrating, and (3.27) again applies. The total potential energy of the system is

$$U^* = AF_F^*\ ,\ F_F^* = \frac{1}{\beta_0}\left[\beta_0 + (\gamma_1 - \frac{3-\kappa}{\kappa+1}\,\gamma)2\epsilon^0\right] F_F\ , \tag{9.29}$$

where F_F is the film free energy associated with a film with a fixed edge, (8.3). The new film free energy F_F^* is associated with a film with a free edge. It must be nonnegative and also less than F_F, and hence

$$-\,\beta_0 < (\gamma_1 - \frac{3-\kappa}{\kappa+1}\,\gamma)2\epsilon^0 < 0 \tag{9.30}$$

It can be shown that for positive β_0, which is presumed throughout this paper, a material volume contracts in the presence of a traction–free surface. For example, $v,_\xi$ of (8.2) is negative. It follows that $\epsilon^0 < 0$ which, together with the second inequality of (9.30), implies that

$$\gamma_1 - \frac{3-\kappa}{\kappa+1}\,\gamma = \frac{2(\kappa-1)}{\kappa+1}\,\gamma_1 - \frac{3-\kappa}{\kappa+1}\,(\gamma_2+\gamma_3) > 0\ . \tag{9.31}$$

The first of (9.30) and (9.28) leads to the conclusion that

$$F(h) = \left[\frac{3-\kappa}{\kappa+1}\gamma-(2\gamma_1+\gamma_2-2\alpha_1-\alpha_2)-(2\beta_1+2\beta_2+\beta_3)(\alpha-\gamma)/\beta\right]F_F/\epsilon\beta_0^2 < 0, \tag{9.32}$$

if

$$1 < \kappa < 3 \tag{9.33}$$

which is a condition for the ordinary theory of elasticity. Since F_F is positive, we have

$$\frac{3-\kappa}{\kappa+1}\gamma - (2\gamma_1+\gamma_2-2\alpha_1-\alpha_2) - (2\beta_1+2\beta_2+\beta_3)(\alpha-\gamma)/\beta < 0 \ . \tag{9.34}$$

A unidirectional tension test, in the z_1–direction, performed on the free film is defined by

$$\epsilon_{11} = \epsilon^0 + e_{11} \ , \ \epsilon_{22} = \epsilon^0 + e_{22} \ , \tag{9.35}$$

$$\langle t_{22}\rangle = \frac{\kappa+1}{\kappa-1}e_{22} + \frac{3-\kappa}{\kappa-1}e_{11} - \left[\frac{(3-\kappa)^2}{\kappa^2-1} + (\gamma_1 - \frac{3-\kappa}{\kappa+1}\gamma)F(h)\right]e_{\alpha\alpha} = 0 \ , \tag{9.36}$$

$$\langle t_{11}\rangle = 2(e_{11}-e_{22}) \ , \tag{9.37}$$

where (9.36) has been used in defining $\langle t_{11}\rangle$.

The vanishing of $\langle t_{22}\rangle$, (9.36), yields

$$e_{22} = -\frac{3-\kappa}{4}G(h) \tag{9.38}$$

where

$$G(h) = \left[1 - \frac{\kappa+1}{2(3-\kappa)}(\gamma_1 - \frac{3-\kappa}{\kappa+1}\gamma)F(h)\right] \Big/ \left[1 - \frac{\kappa+1}{8}(\gamma_1 - \frac{3-\kappa}{\kappa+1}\gamma)F(h)\right] \tag{9.39}$$

It follows from (9.31), (9.32), (9.33) and (9.25) that

$$G(h) > 1 \ , \ \lim_{h\to\infty} G(h) = 1 \ . \tag{9.40}$$

The tension–test result finally becomes

$$\langle t_{11}\rangle = 2\left[1 + \frac{3-\kappa}{4}G(h)\right]e_{11} \ . \tag{9.41}$$

If the film is actually a plate, i.e. $h\to\infty$, the elasticity result is recovered, i.e.

$$\langle t_{11}\rangle = \frac{7-\kappa}{2}e_{11} = 2(1+\nu)\,e_{11} \quad \text{(Elasticity)} \tag{9.42}$$

where $2(1+\nu)$ is to be identified with Young's modulus, as Dim.$\langle t_{11} \rangle = \mu \langle t_{11} \rangle$. The cohesive elasticity solution is

$$\langle t_{11} \rangle = \left\{ 2(1+\nu) + \frac{3-\kappa}{2} \left[G(h) - 1 \right] \right\} e_{11} \quad . \tag{9.43}$$

It follows from (9.40) that the apparent modulus for a film is greater than Young's modulus and, in fact, size dependent. It is widely known that the apparent strengths of some materials are effected by strain gradients. Using a couple–stress theory, which employs only a portion of the strain gradients, Mindlin (1962) was able to show a reduction in stress–concentration factor around a small hole. The above result is consistent with the general belief that increasing strain gradients appear to make some materials stronger. Finally, the analysis of this section has essentially established the groundwork for a new plane–stress theory suitable for films.

10. Concluding Remarks

The inclusion of higher order gradients of displacements in a continuum theory leads to boundary–layer phenomena that are absent in all grade–1 theories of which the ordinary theory of elasticity is a special case. This character may be exploited to facilitate the introduction of nontrival self–equilibrating states, so that surface phenomena of solids may be defined by special displacement fields. If material isotropy is presumed, a the..y must include at least the first three gradients of the displacements before a material constant can be introduced to capture the desired surface phenomena (Mindlin 1965). This is the grade–3 theory of Mindlin and the constant that gives rise to all the surface phenomena was termed the modulus of cohesion by him. It is for this reasc that we have coined the name cohesive elasticity.

The most striking approach in Mindlin's formulatic . is the inclusion of the linear term $\epsilon^2 \beta_0 E_{iijj}$ in the strain energy–density W of (3.17). It ..es rise to a constant triple

stress and leads to the concise energy expression U^* of (3.27). This practice, however, is a matter of convenience in the sense that one could have left the term out of W but introduced the constant triple stress as an initial or residual state. In fact, one could have interpreted the triple stress as a special body force in the sense of (7.28). The important point is that however we interpret it, it is possible to introduce a new constant to generate a self–equilibrating state.

It is, of course, completely legitimate to include the linear term E_{ii} in the elasticity version of strain energy–density. Such a term may also be interpreted as a initial or residual stress. The stress, however, must be completely relieved in the presence of a free boundary, as elasticity equations do not exhibit boundary–layer behavior and, as a result, the exercise leads to no useful results. It is obviously for this reason that a grain with an eigenstrain must interact with a nontrivial surrounding (Mura, 1982, Eshelby, 1957). A nontrivial displacement field is associated with a free grain according to cohesive elasticity. We have analytically defined the meaning of grain boundary energy in Section 7. Unlike ordinary elasticity, cohesive elasticity requires the implementation of corner (boundary) conditions in case the boundary is not a smooth surface. In this connection, it is instructive to recall the presence of a corner force in a plate theory (a grade–2 theory). A notch– or crack–tip and a corner on a grain are among many meaningful problems to be analyzed. Even the meaning of a concentrated force may be re–examined. We have already found that the displacement under a point load is finite.

A grade–2 theory contains couple stress and double stress. It is mathematically more tractable than a grade–3 theory and also exhibits boundary–layer behavior. However, since it is not possible to include a term linear in E_{ijk} in W, it is not possible to introduce a material constant to induce a self–equilibrating state for defining, say, surface free energy. On the other hand, one could introduce an initial or residual double stress, instead of triple stress, to induce a self–equilibrating system. The constant double stress is not a material constant any longer, but is still associated with a special chunk of a

material. This possibility may turn out to be a simpler way of generating the desired surface phenomena.

Higher gradient theories were the popular topics of research of the sixties. Of the many known published results, Mindlin's work consistently placed the most emphasis on relating to real microscopic phenomena of solids. If the goal of today's micromechanics initiative is actually aiming at establishing physically relevant microcontinuum mechanics field theories, then Mindlin's work are indeed of a pioneer nature and should be carefully reexamined.

Every time the word continuum is mentioned, the term constitutive relation follows. One of the commonly asked questions during the sixties was how to determine the so many constants involved in a grade–n theory. It is our belief now that the negative implication of the question was a result of our preoccupation with the term constitutive relation which, in one way or another, is perceived as a response to a uniform sample. The existence of a uniform sample, say, the tension specimen, is a consequence of the assumption that the behavior of a solid is completely characterized by E_{ij}. While the concept of Cauchy stress and stress vector is not dependent on the former presumption, they are actually consistent in that the components of the surface traction vector would turn out to be just the appropriate components of the stress tensor if the latter were defined in terms of a strain energy–density via the stationary potential energy approach of Section 2. It is clear from Section 2 that once higher gradients are included the exact meaning of a Cauchy stress becomes vague and the direct connection between stress and stress vector $\overset{0}{t}$, is gone. In general, there is no uniform sample to speak of anymore. Many new nontrivial problems of a more fundamental nature must first be solved before new test specimens can be devised. The surface free energy, unidirectional test solution, and point load solution (not included in this paper) are but a few of the anticipated test configurations.

No continuum field theories can be a hundred percent physical. For example, it could be argued that Young's modulus should be derived from more fundamental physical

constants, but the "correct" Young's modulus is always measured from a tension specimen which was designed in accordance with the mathematical solution that $\sigma = P/A$ and $\epsilon = \Delta/L$. It is our belief that many, many more P/A formulas must and will be discovered in the future. Finally, in connection with failure, the all important Eshelby tensor must be modified. The tensor itself could be straightforwardly defined by casting the spatial derivatives of W into a divergence free form, but the associated conservation laws remain to be examined. It is even possible to couple the current grade–3 theory with a higher order diffusion equation, the Cahn–Hilliard equation (Cahn and Hilliard, 1957, Gurtin, 1989), so that the modulus of cohesion will depend on concentration and one of the diffusion coefficients will depend on the triple stress. It is perhaps time again to reexamine the role of higher gradients in our pursuit for the understanding of failure of solids.

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